Optimal Multiplexed Erasure Codes for Streaming Messages with Different Decoding Delays

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***Abstract*—This paper considers multiplexing two sequences of messages with two different decoding delays over a packet erasure channel. In each time slot, the source constructs a packet based on the current and previous messages and transmits the packet, which may be erased when the packet travels from the source to the destination. The destination must perfectly recover every source message in the first sequence subject to a decoding delay T_{v} and every source message in the second sequence subject to a shorter decoding delay $T_{\mathrm{u}} \le T_{\mathrm{v}}$. We assume that the channel loss model introduces a burst erasure of a fixed length B on the discrete timeline. Under this channel loss assumption, the capacity region for the case where $T_{\mathrm{v}} \le T_{\mathrm{u}} + B$ was previously solved. In this paper, we fully characterize the capacity region for the remaining case $T_{\mathrm{v}} > T_{\mathrm{u}} + B$.**

I. Introduction

Video streaming applications including video conferencing, virtual reality (VR) and online gaming are expected to dominate 82 percent of the Internet traffic by 2022, up from 75 percent in 2017 [1]. Due to the growing demand in video streaming applications, we are motivated to find effective error correction strategies for low-latency communications. Two main error control schemes have been implemented at the data link layer and the transport layer to alleviate the effect of packet losses on applications that are run over the Internet: Automatic repeat request (ARQ) and forward error correction (FEC). FEC is preferred over ARQ when retransmitting lost packets is costly. For example, retransmitting a Voice-over-IP (VoIP) packet incurs an extra round-trip delay which will result in an overall three-way delay (forward + backward + forward) that may exceed the 150 ms delay recommended by International Telecommunication Union [2]. Given the fact that the three-way propagation delay is at least 200 ms for communication between two diametrically opposite points on the earth's circumference [3], FEC has a clear advantage over ARQ for long-distance low-latency communication.

This paper studies low-latency FEC schemes implemented at the transport layer, where a source packet is either received by the destination without error or dropped by the network (possibly due to unreliable links or network congestion). Since packet erasures often occur in a bursty rather than sparse manner [4,5], we model the connection between the source and the destination as a packet erasure channel that introduces burst erasures. In order to capture the streaming nature of messages and the low-latency requirements, we assume that a source message is encoded into a channel packet in every time slot and a decoding delay constraint T is imposed on every message. If the destination cannot decode a message within T time slots from the time when the message is generated, the message is considered lost. Characterizing the maximum achievable rates for statistical models that generate burst erasures such as the well-known Gilbert-Elliott (GE) channel [6,7] and the Fritchman channel [8] seems intractable due to the delay constraint and the fact that those statistical models are not memoryless. Therefore, Martinian and Sundberg [9] have instead characterized the capacity, i.e., maximum coding rate, for a simpler deterministic model where a burst erasure of length B is introduced on the discrete timeline and every message must be perfectly recovered at the destination with a decoding delay of T time slots. They proposed a streaming code that not only achieves the capacity $\frac{T}{T+B}$ for the deterministic model, but also can significantly outperform traditional FEC schemes for the GE channel. Various generalizations of the packet erasure model and the streaming codes in [9] have been proposed in [10]–[14].

Note that an application may consist of multiple types of data streams (video, audio, text, etc.), and also within a single data stream such as video there are different subsets of data that have different delivery deadlines. Moreover, multiplexing streams of different types has been implemented in the QUIC transport protocol for reducing the latency of Google Search and YouTube [15]. Therefore, Badr et al. [16] extended the study of single-stream codes in [9] and initiated the study of streaming codes which multiplex a stream of urgent messages with a stringent delay constraint and a stream of less-urgent messages with a less stringent delay constraint. Simulation results in [16, Sec. VIII] demonstrate that using multiplexed streaming codes can significantly outperform concatenating multiple single-stream codes for the GE channel. In [16], every urgent message and every less-urgent message generated in a time slot have to be decoded within T_{u} and T_{v} time slots respectively from the time when the messages are generated where $T_{\mathrm{u}} \le T_{\mathrm{v}}$. Similar to the single-stream case, we assume that the channel introduces a burst erasure of length B on the discrete timeline and define the capacity region to be the set of rate pairs $(R_{\mathrm{v}}, R_{\mathrm{u}})$ which are supported by streaming codes that correct any length-B burst erasure where R_{v} and R_{u} denote the rates of the less-urgent stream and urgent stream respectively. For the case $T_{\mathrm{u}} \le T_{\mathrm{v}} \le T_{\mathrm{u}} + B$, systematic streaming codes have been proposed in [16] to achieve the capacity region. However, it is unclear whether the capacity region can be achieved by the multiplexed streaming codes proposed in [16] in general. Therefore, we are motivated to characterize the capacity region in the remaining cases.

The rest of the paper is organized as follows. The notation in this paper is explained in the next section. Section III presents the formulation of multiplexed streaming codes for the deterministic erasure model and narrows our focus to the only open case $T_{\mathrm{v}} > T_{\mathrm{u}} + B$. Section IV formally defines the capacity region. Section V presents our main result — the capacity region for the open case. Section VI provides a proof sketch for the achievability part. Section VII concludes this paper.

II. Notation

We use $\mathbf{1}\{\mathcal{E}\}$ to denote the indicator function of an event $\mathcal{E}$. The sets of non-negative integers and non-negative real numbers are denoted by $\mathbb{Z}_+$ and $\mathbb{R}_+$ respectively. All the elements of any matrix considered in this paper are taken from a common finite field $\mathbb{F}$. The set of k-dimensional row vectors over $\mathbb{F}$ is denoted by $\mathbb{F}^k$, and the set of $k \times n$ matrices over $\mathbb{F}$ is denoted by $\mathbb{F}^{k\times n}$. A row vector in $\mathbb{F}^k$ is denoted by $\mathbf{a} \triangleq [a_0\ a_1\ \ldots\ a_{k-1}]$. The k-dimensional identity matrix is denoted by $\mathbf{I}_k$ and the $L \times B$ all-zero matrix is denoted by $\mathbf{0}^{L\times B}$. We call a matrix $\mathbf{V}^{L\times B}$ an $L \times B$ parity matrix of a systematic maximum-distance separable (MDS) $(L + B, L)$-code if any L columns of $[\mathbf{I}_L\ \mathbf{V}^{L\times B}] \in \mathbb{F}^{L\times(L+B)}$ are independent. A systematic MDS $(L+B, L)$-code always exists as long as $|\mathbb{F}| \ge L + B$ [17].

III. Multiplexed Streaming Codes for Channels with Burst Erasures

The source wants to simultaneously send a sequence of length-k_{u} packets $\mathbf{u}^\infty \triangleq \{\mathbf{u}_i\}_{i=0}^\infty$ with decoding delay T_{u} and a sequence of length-k_{v} packets $\mathbf{v}^\infty \triangleq \{\mathbf{v}_i\}_{i=0}^\infty$ with decoding delay $T_{\mathrm{v}} \ge T_{\mathrm{u}}$ to the destination, where k_{u} and k_{v} denote the sizes of each urgent packet $\mathbf{u}_i$ and each less-urgent packet $\mathbf{v}_i$ respectively. Each $\mathbf{u}_i$ is an element in $\mathbb{F}^{k_{\mathrm{u}}}$ and each $\mathbf{v}_i$ is an element in $\mathbb{F}^{k_{\mathrm{v}}}$ where $\mathbb{F}$ is some finite field. In each time slot $i \in \mathbb{Z}_+$, the source packets $\mathbf{v}_i$ and $\mathbf{u}_i$ are encoded into a length-n packet $\mathbf{x}_i \in \mathbb{F}^n$ to be transmitted to the destination through an erasure channel, and the destination receives $\mathbf{y}_i \in \mathbb{F}^n \cup \{*\}$ where $\mathbf{y}_i$ equals either $\mathbf{x}_i$ or the erasure symbol '$*$'. The fractions k_{u}/n and k_{v}/n specify the rates of the urgent and less-urgent streams respectively. The urgent and less-urgent streams are subject to the delay constraints of T_{u} and T_{v} time slots respectively, meaning that the destination must produce an estimate of $\mathbf{u}_i$, denoted by $\hat{\mathbf{u}}_i$, upon receiving $\mathbf{y}_{i+T_{\mathrm{u}}}$ and produce an estimate of $\mathbf{v}_i$, denoted by $\hat{\mathbf{v}}_i$, upon receiving $\mathbf{y}_{i+T_{\mathrm{v}}}$. We assume that the channel introduces a burst erasure of length B on the discrete timeline as in [16].

We assume without loss of generality (wlog) that $T_{\mathrm{v}} \ge B$ or otherwise a burst erasure of length B starting from time i would prevent the destination to recover $\mathbf{u}_i$ and $\mathbf{v}_i$ by time $i + T_{\mathrm{v}}$. If the channel is noiseless where $B = 0$, no coding is needed to asymptotically achieve all the rate pairs $(k_{\mathrm{v}}/n, k_{\mathrm{u}}/n)$ on the boundary of the capacity region that satisfy $k_{\mathrm{u}}/n + k_{\mathrm{v}}/n = 1$. Therefore, we assume wlog that $B \ge 1$. For the case $T_{\mathrm{u}} < B$, it can be observed that a burst erasure of length B starting from time i would prevent the destination to recover $\mathbf{u}_i$ by time $i+T_{\mathrm{u}}$. Consequently, no rate pair $(k_{\mathrm{v}}/n, k_{\mathrm{u}}/n)$ with $k_{\mathrm{u}}/n > 0$ is achievable, which implies that the capacity region reduces to the interval $[0, \mathrm{C}(T_{\mathrm{v}}, B)]$ on the horizontal axis where

$$\mathrm{C}(T, B) \triangleq \frac{T}{T + B} \tag{1}$$

is the capacity achieved by streaming codes with delay T that correct any length-B burst erasure [9, Th. 1 and Th. 2] (see also [14, Sec. III-C]). Since the case $T_{\mathrm{u}} < B$ degenerates the multiplexing problem to the previously known single-stream problem described above, and we assume wlog that $T_{\mathrm{u}} \ge B$. For the case $T_{\mathrm{u}} = T_{\mathrm{v}}$, since the urgent and less-urgent source packets can be viewed as single-stream source packets with delay T_{u}, any rate pair $(k_{\mathrm{v}}/n, k_{\mathrm{u}}/n)$ must satisfy $k_{\mathrm{u}}/n + k_{\mathrm{v}}/n \le \mathrm{C}(T_{\mathrm{u}}, B)$ (recall that the capacity of the single-stream problem equals $\mathrm{C}(T, B)$ by [9]). In addition, the boundary of the capacity region $k_{\mathrm{u}}/n + k_{\mathrm{v}}/n = \mathrm{C}(T_{\mathrm{u}}, B)$ can be asymptotically achieved by partitioning each source packet of an optimal code with rate $\mathrm{C}(T_{\mathrm{u}}, B)$ into an urgent source packet and a less-urgent source packet. Consequently, the case $T_{\mathrm{u}} = T_{\mathrm{v}}$ degenerates the multiplexing problem to a single-stream problem described above, and we assume wlog that $T_{\mathrm{v}} > T_{\mathrm{u}}$. Summarizing the aforementioned assumptions, we assume in the rest of the paper that

$$T_{\mathrm{v}} > T_{\mathrm{u}} \ge B \ge 1. \tag{2}$$

Any condition that does not satisfy (2) leads to the aforementioned known results. For the special case where

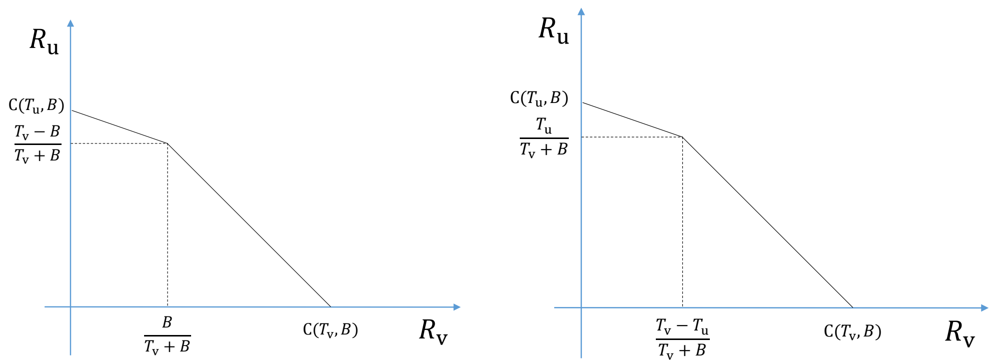
$$1 \le B \le T_{\mathrm{u}} < T_{\mathrm{v}} \le T_{\mathrm{u}} + B, \tag{3}$$

systematic streaming codes have been proposed in [16] to achieve the capacity region, which is the set of rate pairs $(R_{\mathrm{v}}, R_{\mathrm{u}})$ satisfying $\left(1 + \frac{T_{\mathrm{u}}+B-T_{\mathrm{v}}}{T_{\mathrm{u}}}\right) R_{\mathrm{v}} + \frac{R_{\mathrm{u}}}{\mathrm{C}(T_{\mathrm{u}},B)} \le 1$ and $R_{\mathrm{v}} + R_{\mathrm{u}} \le \mathrm{C}(T_{\mathrm{v}}, B)$ as illustrated in Figure 1(a). In addition, other systematic streaming codes have been proposed in [16] to achieve two different rate regions for the cases $T_{\mathrm{u}} + B < T_{\mathrm{v}} < T_{\mathrm{u}} + 2B$ and $T_{\mathrm{v}} \ge T_{\mathrm{u}} + 2B$ respectively, denoted by $\mathcal{R}_{\{T_{\mathrm{u}}+B<T_{\mathrm{v}}<T_{\mathrm{u}}+2B\}}$ and $\mathcal{R}_{\{T_{\mathrm{v}}\ge T_{\mathrm{u}}+2B\}}$ respectively. In particular, if only systematic streaming codes are allowed, $\mathcal{R}_{\{T_{\mathrm{v}}\ge T_{\mathrm{u}}+2B\}}$ was shown in [16] to be the largest.

IV. Capacity Region

Since (2) is assumed and the capacity region for case (3) was proved in [16], this paper solves the only remaining case

$$T_{\mathrm{v}} > T_{\mathrm{u}} + B. \tag{4}$$



(a) Case $T_\mathrm{u} < T_\mathrm{v} \leq T_\mathrm{u} + B$

(b) Case $T_\mathrm{v} > T_\mathrm{u} + B$

Fig. 1. Capacity region

Before presenting the main result of this paper, we formally define multiplexed streaming codes and the capacity region.

Definition 1: An $(n, k_\mathrm{v}, k_\mathrm{u}, T_\mathrm{v}, T_\mathrm{u})_\mathbb{F}$-streaming code consists of

1) A sequence of length-k_v less-urgent source packets $\mathbf{v}^\infty$.
2) A sequence of length-k_u urgent source packets $\mathbf{u}^\infty$.
3) An encoder $f_i : \mathbb{F}^{k_\mathrm{u}+k_\mathrm{v}} \times \ldots \times \mathbb{F}^{k_\mathrm{u}+k_\mathrm{v}} \to \mathbb{F}^n$ for each $i \in \mathbb{Z}_+$ where $\mathbf{x}_i = f_i((\mathbf{u}_0, \mathbf{v}_0), (\mathbf{u}_1, \mathbf{v}_1), \ldots, (\mathbf{u}_i, \mathbf{v}_i))$.
4) A decoder $\varphi^{(\mathrm{v})}_{i+T_\mathrm{v}} : \mathbb{F}^n \cup \{*\} \times \ldots \times \mathbb{F}^n \cup \{*\} \to \mathbb{F}^{k_\mathrm{v}}$ for each $i \in \mathbb{Z}_+$ where $\hat{\mathbf{v}}_i = \varphi^{(\mathrm{v})}_{i+T_\mathrm{v}}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{i+T_\mathrm{v}})$.
5) A decoder $\varphi^{(\mathrm{u})}_{i+T_\mathrm{u}} : \mathbb{F}^n \cup \{*\} \times \ldots \times \mathbb{F}^n \cup \{*\} \to \mathbb{F}^{k_\mathrm{u}}$ for each $i \in \mathbb{Z}_+$ where $\hat{\mathbf{u}}_i = \varphi^{(\mathrm{u})}_{i+T_\mathrm{u}}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{i+T_\mathrm{u}})$.

In addition, the code is said to be *systematic* if $\mathbf{x}_i = [\mathbf{v}_i\ \mathbf{u}_i\ \mathbf{a}_i]$ for some $\mathbf{a}_i \in \mathbb{F}^{n-k_\mathrm{v}-k_\mathrm{u}}$ at each time $i \in \mathbb{Z}_+$.

Definition 2: An erasure sequence is a binary sequence $\mathbf{e} \triangleq \{e_i\}_{i=0}^\infty$ where $e_i = \mathbf{1}\{\text{erasure occurs at time } i\}$. If $\sum_{i=0}^\infty e_i = B$ with all the 1's occupying consecutive positions, $\mathbf{e}$ is also called a *B-erasure sequence*. The set of B-erasure sequences is denoted by Ω_B. Similarly, for any $n \geq B$, a length-n binary sequence denoted by $e^n \triangleq \{e_i\}_{i=0}^{n-1}$ is called a B-erasure sequence if e^n satisfies $\sum_{i=0}^{n-1} e_i = B$ with all the 1's occupying consecutive positions. The set of length-n B-erasure sequences is denoted by Ω_B^n.

Definition 3: The mapping $g_n : \mathbb{F}^n \times \{0, 1\} \to \mathbb{F}^n \cup \{*\}$ of the erasure channel is defined as

$$g_n(\mathbf{x}, e) = \begin{cases} \mathbf{x} & \text{if } e = 0, \\ * & \text{if } e = 1. \end{cases} \tag{5}$$

For any erasure sequence $\mathbf{e}$ and any $(n, k_\mathrm{v}, k_\mathrm{u}, T_\mathrm{v}, T_\mathrm{u})_\mathbb{F}$-streaming code, $\mathbf{y}_i = g_n(\mathbf{x}_i, e_i)$ holds for each $i \in \mathbb{Z}_+$.

Definition 4: An $(n, k_\mathrm{v}, k_\mathrm{u}, T_\mathrm{v}, T_\mathrm{u})_\mathbb{F}$-streaming code is said to correct a B-erasure sequence $\mathbf{e}$ if $[\hat{\mathbf{u}}_i\ \hat{\mathbf{v}}_i] = [\mathbf{u}_i\ \mathbf{v}_i]$ holds for all $i \in \mathbb{Z}_+$ and all $[\mathbf{u}_i\ \mathbf{v}_i] \in \mathbb{F}^{k_\mathrm{u}+k_\mathrm{v}}$, where $\hat{\mathbf{u}}_i$ and $\hat{\mathbf{v}}_i$ are determined by $\{(\mathbf{u}_\ell, \mathbf{v}_\ell, e_\ell)\}_{\ell=0}^i$ due to Definitions 1 and 3.

Definition 5: A rate pair $(R_\mathrm{v}, R_\mathrm{u}) \in \mathbb{R}_+^2$ is said to be $(T_\mathrm{v}, T_\mathrm{u}, B)$-achievable if there exists an $(n, k_\mathrm{v}, k_\mathrm{u}, T_\mathrm{v}, T_\mathrm{u})_\mathbb{F}$-streaming code which corrects any B-erasure sequence such that $\frac{k_\mathrm{v}}{n} \geq R_\mathrm{v}$ and $\frac{k_\mathrm{u}}{n} \geq R_\mathrm{u}$.

The following corollary is a direct consequence of Definition 5 and the existing single-stream result [9, Th. 2] (see also [14, Th. 1])) stated as follows: Suppose $T \geq B \geq 1$. Then, there exists a streaming code with rate $\mathrm{C}(T, B)$ which guarantees the recovery of every streaming message with delay T when the channel is subject to any length-B burst erasure on the discrete timeline.

Corollary 1 ([9, Th. 2]): The rate pairs $(\mathrm{C}(T_\mathrm{v}, B), 0)$ and $(0, \mathrm{C}(T_\mathrm{u}, B))$ are $(T_\mathrm{v}, T_\mathrm{u}, B)$-achievable.

Definition 6: Fix any $(T_\mathrm{v}, T_\mathrm{u}, B)$ that satisfies (2). The $(T_\mathrm{v}, T_\mathrm{u}, B)$-capacity region, denoted by $\mathcal{C}_{T_\mathrm{v}, T_\mathrm{u}, B}$, is the closure of the set of $(T_\mathrm{v}, T_\mathrm{u}, B)$-achievable rate pairs.

The following convexity statement regarding $\mathcal{C}_{T_\mathrm{v}, T_\mathrm{u}, B}$ will help us simplify the achievability proof of our main result.

Corollary 2 ([18, Appendix A]): For any $(T_\mathrm{v}, T_\mathrm{u}, B)$ that satisfies (2), $\mathcal{C}_{T_\mathrm{v}, T_\mathrm{u}, B}$ is convex.

V. Main Result

The following theorem is the main result of this paper, which states the capacity region in terms of the single-stream capacity function $\mathrm{C}(\cdot, \cdot)$ as defined in (1).

Theorem 1: Fix any $(T_\mathrm{v}, T_\mathrm{u}, B)$ satisfying (2) and (4). Let

$$\mathcal{R}_{\{T_\mathrm{v} > T_\mathrm{u}+B\}} \triangleq \left\{ (R_\mathrm{v}, R_\mathrm{u}) \in \mathbb{R}_+^2 \,\middle|\, \begin{aligned} & R_\mathrm{v} + \tfrac{R_\mathrm{u}}{\mathrm{C}(T_\mathrm{u}, B)} \leq 1, \\ & R_\mathrm{v} + R_\mathrm{u} \leq \mathrm{C}(T_\mathrm{v}, B) \end{aligned} \right\}$$

which is as illustrated in Figure 1(b). Then,

$$\mathcal{C}_{T_\mathrm{v}, T_\mathrm{u}, B} = \mathcal{R}_{\{T_\mathrm{v} > T_\mathrm{u}+B\}}.$$

The complete proof of Theorem 1 is contained in [18]. The key step in the achievability proof is achieving the non-trivial corner point of the capacity region through using a multiplexed streaming code constructed by superimposing two single-stream codes, and a proof sketch is provided in the next section. The key step in the converse proof is obtaining the genie-aided outer bound $R_\mathrm{v} + \frac{R_\mathrm{u}}{\mathrm{C}(T_\mathrm{u}, B)} \leq 1$ when the channel is subject to a periodic erasure pattern where each period consists of a length-B burst erasure followed by a length-T_u noiseless duration. The main challenge is to provide the genie the least amount of information so that both streams can be perfectly recovered at the destination. The converse proof is omitted due to space limitation.

Remark 1: Consider the case $T_{\mathrm{v}} \geq T_{\mathrm{u}} + 2B$. It was shown in [16, Th. 1] that systematic streaming codes achieves $\mathcal{R}_{\{T_{\mathrm{v}} > T_{\mathrm{u}}+B\}}$. Therefore, systematic streaming codes are sufficient to achieve the capacity region by Theorem 1.

Remark 2: Consider the case $T_{\mathrm{u}} + B < T_{\mathrm{v}} < T_{\mathrm{u}} + 2B$. The systematic streaming codes proposed in [16, Th. 1] cannot achieve the non-trivial corner point $\left(\frac{T_{\mathrm{v}}-T_{\mathrm{u}}}{T_{\mathrm{v}}+B}, \frac{T_{\mathrm{u}}}{T_{\mathrm{v}}+B}\right)$ of $\mathcal{C}_{T_{\mathrm{v}},T_{\mathrm{u}},B}$. On the other hand, our achievability proof proposes a *non-systematic* streaming code that achieve the non-trivial corner point. It remains open whether systematic streaming codes are sufficient to achieve the capacity region.

VI. Achievability Proof of Main Result

The achievability proof of Theorem 1 consists of two steps. The first step involves constructing a multiplexed block code which corrects any B-erasure sequence. The second step involves constructing a multiplexed streaming code which corrects any B-erasure sequence by periodically interleaving the multiplexed block code. The formal definitions and existing results related to multiplexed block codes and periodic interleaving are presented as follows.

Definition 7: An $(n, k_{\mathrm{v}}, k_{\mathrm{u}}, T_{\mathrm{v}}, T_{\mathrm{u}})_{\mathbb{F}}$-block code consists of

1) A vector of k_{v} less-urgent source symbols in $\mathbb{F}$ denoted by $\vec{v} \triangleq \left[v[0]\ v[1]\ \ldots\ v[k_{\mathrm{v}}-1]\right]$.
2) A vector of k_{u} urgent source symbols in $\mathbb{F}$ denoted by $\vec{u} \triangleq \left[u[0]\ u[1]\ \ldots\ u[k_{\mathrm{u}}-1]\right]$.
3) A generator matrix $\mathbf{G} \in \mathbb{F}^{(k_{\mathrm{v}}+k_{\mathrm{u}})\times n}$. The codeword is generated according to $\left[x[0]\ \ldots\ x[n-1]\right] = [\vec{v}\ \vec{u}]\,\mathbf{G}$.
4) A decoder $\varphi^{(\mathrm{v})}_{i+T_{\mathrm{v}}} : \mathbb{F}\cup\{*\} \times \ldots \times \mathbb{F}\cup\{*\} \to \mathbb{F}$ for each $i \in \{0, 1, \ldots, k_{\mathrm{v}}-1\}$ where
$$\hat{v}[i] = \varphi^{(\mathrm{v})}_{i+T_{\mathrm{v}}}(y[0], y[1], \ldots, y[\min\{i+T_{\mathrm{v}}, n-1\}]).$$
5) A decoder $\varphi^{(\mathrm{u})}_{i+T_{\mathrm{u}}} : \mathbb{F}\cup\{*\} \times \ldots \times \mathbb{F}\cup\{*\} \to \mathbb{F}$ for each $i \in \{0, 1, \ldots, k_{\mathrm{u}}-1\}$ where
$$\hat{u}[i] = \varphi^{(\mathrm{u})}_{i+T_{\mathrm{u}}}(y[0], y[1], \ldots, y[\min\{i+T_{\mathrm{u}}, n-1\}]).$$

The following definition concerns the error-correcting capability of a block code.

Definition 8: An $(n, k_{\mathrm{v}}, k_{\mathrm{u}}, T_{\mathrm{v}}, T_{\mathrm{u}})_{\mathbb{F}}$-block code is said to correct a B-erasure sequence $e^n \in \Omega_B^n$ if the following holds: Let $y[i] = g_1(x[i], e_i)$ be the symbol received by the destination at time i for each $i \in \{0, 1, \ldots, n-1\}$ where g_1 is defined in (5). Then, $\hat{v}[i] = v[i]$ holds for all $i \in \{0, 1, \ldots, k_{\mathrm{v}}-1\}$ and all $v[i] \in \mathbb{F}$, and $\hat{u}[i] = u[i]$ holds for all $i \in \{0, 1, \ldots, k_{\mathrm{u}}-1\}$ and all $u[i] \in \mathbb{F}$, where $\hat{v}[i]$ and $\hat{u}[i]$ are determined by $\{(u[\ell], v[\ell], e_\ell)\}_{\ell=0}^{i}$ due to Definition 7.

The following lemma implies that constructing a streaming code which corrects any length-B burst erasure is not more difficult than constructing a block code which corrects any length-B burst erasure. The proof follows the standard argument of interleaving a block code into a streaming code by means of periodic interleaving [19] (see also [9, Sec. IV-A]).

Lemma 3 ([18, Lemma 3]): Given an $(n, k_{\mathrm{v}}, k_{\mathrm{u}}, T_{\mathrm{v}}, T_{\mathrm{u}})_{\mathbb{F}}$-block code which corrects any B-erasure sequence, we can construct an $(n, k_{\mathrm{v}}, k_{\mathrm{u}}, T_{\mathrm{v}}, T_{\mathrm{u}})_{\mathbb{F}}$-streaming code which corrects any B-erasure sequence.

Example 1: Suppose we are given a $(5, 2, 1, 3, 2)_{\mathbb{F}}$-block code which corrects any length-2 busrt erasure with generator matrix $\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. Let $\{\mathbf{v}_i\}_{i\in\mathbb{Z}_+}$ and $\{\mathbf{u}_i\}_{i\in\mathbb{Z}_+}$ be the messages of the less-urgent and urgent streams respectively where $\mathbf{v}_i = \left[v_i[0]\ v_i[1]\right] \in \mathbb{F}^2$ and $\mathbf{u}_i = u_i[0] \in \mathbb{F}$. From time $i-2$ to $i+4$, the symbols yielded by the $(5, 2, 1, 3, 2)_{\mathbb{F}}$-streaming code constructed by interleaving the block code according to Lemma 3 are shown in Table I. The symbols in Table I which are highlighted in the same color diagonally (in ↘ direction) are encoded using the same $(5, 2, 1, 3, 2)_{\mathbb{F}}$-block code. Given the fact that the block code corrects any length-2 burst erasure, we can see from Table I that $\left[v_i[0]\ v_i[1]\right]$ and $u_i[0]$ can be perfectly recovered by time $i+3$ and time $i+2$ respectively as long as the erasure sequence is taken from Ω_2^5. ■

Lemma 3 reduces the problem of finding streaming codes which correct any B-erasure sequence to the problem of finding block codes which correct any B-erasure sequence. We will construct high-rate block codes by superimposing the codewords of two single-stream block codes, and therefore we need the following definition of a single-stream block code.

Definition 9: An $(n, k, T)_{\mathbb{F}}$*-block code* is an $(n, k, 0, T, 0)_{\mathbb{F}}$-block code and is said to correct a B-erasure sequence e^n if the equivalent $(n, k, 0, T, 0)_{\mathbb{F}}$-block code corrects e^n.

By Definition 9, an $(n, k, T)_{\mathbb{F}}$-block code ignores the urgent stream of messages by setting the message size for the urgent stream to zero. The following lemma is a restatement of an existing construction [9, Sec. III] (see also [14, Sec. III]) of an $(n, k, T)_{\mathbb{F}}$-block code with rate $k/n = \mathrm{C}(T, B)$ which corrects any length-B burst erasure.

Lemma 4: Suppose $T \geq B \geq 1$ and let $k \triangleq T$ and $n \triangleq k + B$. Fix any $\mathbb{F}$ with $|\mathbb{F}| \geq T$ and let $\mathbf{P}$ denote the parity matrix of a systematic MDS $(T, T-B)$-code. Then, the $(n, k, T)_{\mathbb{F}}$-block code with rate $k/n = \mathrm{C}(T, B)$ and generator matrix $\mathbf{G} \triangleq \left[\begin{array}{c|c} \mathbf{I}_T & \begin{array}{c}\mathbf{I}_B \\ \mathbf{P}\end{array} \end{array}\right]$ corrects any length-B burst erasure.

Achievability proof of Theorem 1: Our goal is to show that $\mathcal{C}_{T_{\mathrm{v}},T_{\mathrm{u}},B} \supseteq \mathcal{R}_{\{T_{\mathrm{v}} > T_{\mathrm{u}}+B\}}$. By Corollary 2, it suffices to show that the four corner points of $\mathcal{R}_{\{T_{\mathrm{v}} > T_{\mathrm{u}}+B\}}$ are $(T_{\mathrm{v}}, T_{\mathrm{u}}, B)$-achievable. Since the corner points $(0, 0)$, $(\mathrm{C}(T_{\mathrm{v}}, B), 0)$ and $(0, \mathrm{C}(T_{\mathrm{u}}, B))$ are $(T_{\mathrm{v}}, T_{\mathrm{u}}, B)$-achievable by Corollary 1, it suffices to show that the remaining corner point $\left(\frac{T_{\mathrm{v}}-T_{\mathrm{u}}}{T_{\mathrm{v}}+B}, \frac{T_{\mathrm{u}}}{T_{\mathrm{v}}+B}\right)$ is $(T_{\mathrm{v}}, T_{\mathrm{u}}, B)$-achievable. To this end, we let $k_{\mathrm{v}} \triangleq T_{\mathrm{v}} - T_{\mathrm{u}} > 0$, $k_{\mathrm{u}} \triangleq T_{\mathrm{u}}$ and $n \triangleq T_{\mathrm{v}} + B$, and will construct an $(n, k_{\mathrm{v}}, k_{\mathrm{u}}, T_{\mathrm{v}}, T_{\mathrm{u}})_{\mathbb{F}}$-block code which corrects any length-B burst erasure. Fix any $\mathbb{F}$ with $|\mathbb{F}| \geq \max\{T_{\mathrm{u}}, T_{\mathrm{v}} - T_{\mathrm{u}}\}$. Let $\mathbf{V}$ and $\mathbf{U}$ be the parity matrices of a systematic MDS $(T_{\mathrm{v}}-T_{\mathrm{u}}, T_{\mathrm{v}}-T_{\mathrm{u}}-B)$-code and a systematic

Time / Symbol	$i-2$	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
0	$v_{i-2}[0]$	$v_{i-1}[0]$	$v_i[0]$	$v_{i+1}[0]$	$v_{i+2}[0]$	$v_{i+3}[0]$	$v_{i+4}[0]$
1	$v_{i-2}[1]$	$v_{i-1}[1]$	$v_i[1]$	$v_{i+1}[1]$	$v_{i+2}[1]$	$v_{i+3}[1]$	$v_{i+4}[1]$
2	$u_{i-2}[0]$	$u_{i-1}[0]$	$u_i[0]$	$u_{i+1}[0]$	$u_{i+2}[0]$	$u_{i+3}[0]$	$u_{i+4}[0]$
3	$\ddots$	$\ddots$	$\ddots$	$v_{i-2}[0]+u_i[0]$	$v_{i-1}[0]+u_{i+1}[0]$	$v_i[0]+u_{i+2}[0]$	$\ddots$
4	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$v_{i-1}[1]+u_i[0]$	$v_i[1]+u_{i+1}[0]$	$v_{i+1}[1]+u_{i+2}[0]$

TABLE I

SYMBOLS YIELDED BY A $(5,2,1,3,2)_{\mathbb{F}}$-STREAMING CODE THROUGH INTERLEAVING A $(5,2,1,3,2)_{\mathbb{F}}$-BLOCK CODE.

MDS $(T_{\mathrm{u}}, T_{\mathrm{u}}-B)$-code respectively, and let

$$\mathbf{G} \triangleq \left[\begin{array}{c|c|cc} \mathbf{I}_{T_{\mathrm{v}}-T_{\mathrm{u}}} & \begin{matrix}\mathbf{I}_B \\ \mathbf{V}\end{matrix} & \multicolumn{2}{c}{\mathbf{0}^{(T_{\mathrm{v}}-T_{\mathrm{u}})\times B}} \\ \hline \mathbf{0}^{T_{\mathrm{u}}\times(T_{\mathrm{v}}-T_{\mathrm{u}})} & \begin{matrix}\mathbf{I}_B \\ \mathbf{0}^{(T_{\mathrm{u}}-B)\times B}\end{matrix} & \begin{matrix}\mathbf{0}^{B\times(T_{\mathrm{u}}-B)} \\ \mathbf{I}_{T_{\mathrm{u}}-B}\end{matrix} & \begin{matrix}\mathbf{I}_B \\ \mathbf{U}\end{matrix} \end{array}\right]$$

be the generator matrix of the $(n, k_{\mathrm{v}}, k_{\mathrm{u}}, T_{\mathrm{v}}, T_{\mathrm{u}})_{\mathbb{F}}$-block code. The intuition behind the construction of $\mathbf{G}$ is to superimpose the codeword generated from the less-urgent symbols $\vec{x}_v \triangleq \vec{v}\left[\begin{array}{c|c} \mathbf{I}_{T_{\mathrm{v}}-T_{\mathrm{u}}} & \begin{matrix}\mathbf{I}_B \\ \mathbf{V}\end{matrix}\end{array}\right]$ and the codeword generated from the urgent symbols $\vec{x}_u \triangleq \vec{u}\left[\begin{array}{c|c} \mathbf{I}_{T_{\mathrm{u}}} & \begin{matrix}\mathbf{I}_B \\ \mathbf{U}\end{matrix}\end{array}\right]$ such that the two streams interfere with each other in the resultant codeword at B consecutive positions. Since both codewords $\vec{x}_v$ and $\vec{x}_u$ correct any length-B burst erasure by Lemma 4, we claim that the superimposed codeword

$$[\vec{v}\ \vec{u}]\mathbf{G} = \begin{array}{c} [- - - - \ \vec{x}_v \ - - - - \quad \mathbf{0}^{(T_{\mathrm{v}}-T_{\mathrm{u}})\times B}] \\ + \\ [\mathbf{0}^{T_{\mathrm{u}}\times(T_{\mathrm{v}}-T_{\mathrm{u}})} \quad - - - - \ \vec{x}_u \ - - - -] \end{array}$$

also corrects any length-B burst erasure, where the last B symbols of $\vec{x}_v$ interfere with the first B symbols of $\vec{x}_u$. A rigorous proof of the claim is contained in [18, Sec. III-B]. In other words, the $(n, k_{\mathrm{v}}, k_{\mathrm{u}}, T_{\mathrm{v}}, T_{\mathrm{u}})_{\mathbb{F}}$-block code with generator matrix $\mathbf{G}$ corrects any length-B burst erasure, which together with Lemma 3 implies that $\left(\frac{T_{\mathrm{v}}-T_{\mathrm{u}}}{T_{\mathrm{v}}+B}, \frac{T_{\mathrm{u}}}{T_{\mathrm{v}}+B}\right)$ is $(T_{\mathrm{v}}, T_{\mathrm{u}}, B)$-achievable (cf. Definition 5). ∎

VII. CONCLUDING REMARKS

The capacity region has been shown for case (4) under assumption (2), which together with the existing results stated in Section III implies the full characterization of the capacity region for all $(T_{\mathrm{v}}, T_{\mathrm{u}}, B)$. It remains open whether systematic streaming codes are capacity-achieving when $T_{\mathrm{u}} + B < T_{\mathrm{v}} < T_{\mathrm{u}} + 2B$. Theorem 1 is readily generalized to the following deterministic model that generates multiple burst erasures as explained in [16, Remark 1]: The channel introduces multiple burst erasures on the discrete timeline where the length of each burst does not exceed B and the length of the guard space between two adjacent bursts is at least T_{v}. Future work may generalize the capacity region to the erasure model which introduces both burst and arbitrary erasures [14,20].

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