Lattice Network Coding via Signal Codes

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Compute-and-Forward
Nazer & Gastpar (2006)
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\[ y_3 = h_{13} x_1 + h_{23} x_2 + z_3 \]

\[ y_4 = h_{14} x_1 + h_{24} x_2 + z_4 \]
\[ w_3 = a_{13}w_1 + a_{23}w_2 \]
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Compute-and-Forward
Nazer & Gastpar (2006)

\[ w_3 = a_{13} w_1 + a_{23} w_2 \]

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Compute-and-Forward
Nazer & Gastpar (2006)

$w_3 = a_{13}w_1 + a_{23}w_2$

$w_4 = a_{14}w_1 + a_{24}w_2$

$y_5 = h_{35}x_3 + h_{45}x_4 + z_5$
Compute-and-Forward
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\[ w_3 = a_{13}w_1 + a_{23}w_2 \]

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\[ w_5 = a_{35}w_3 + a_{45}w_4 \]

\[ y_5 = h_{35}x_3 + h_{45}x_4 + z_5 \]
Introduction

Lattice network coding
- Lattice-based compute-and-forward (Nazer & Gastpar, 2006)
- Linear physical-layer network coding
- R. Zamir, “Lattices are everywhere”

Related approaches
- Narayanan-Wilson-Sprintson (2007)
- Nam-Chung-Lee (2008)
Nazer & Gastpar’s approach

- Lattice partitions based on Erez-Zamir’s construction
- **Main result**: achievable rates for one-hop networks
- CSI only at the receivers but **not** at the transmitters
- **However**: asymptotically long block length and unbounded complexity

Our goal: practical codes for compute-and-forward

- Finite block length and low complexity
- Example: wireless fading channel with short coherence time

Related work

Ordentlich et al ISIT 2011, Hern & Narayanan ISIT 2011
Our Previous Work

“An algebraic approach to physical-layer network coding” (ISIT 2010)
- Lattice partition → a module structure on the message space
- Fundamental theorem of finitely generated modules over a PID
- Generalized constructions over complex numbers
- Allows working with Eisenstein as well as Gaussian integers

“Design criteria for lattice network coding” (CISS 2011)
- Choice of receiver parameters ($\alpha$, $a$)
- Shortest vector problem
- Upper bound on error probability for hypercube shaping

For more details:
“An algebraic approach to physical-layer network coding” (submitted to IEEE Transactions on Information Theory, July 2011)
This Work

Signal codes (Shalvi, Sommer & Feder 2003)

- Convolutional lattice codes
- Reasonably high coding gain
- Efficient encoding and decoding methods
- Relatively short packet length
- Issue: how to deal with the convolution tail
- Approach: send side information to reconstruct the tail

Contributions

- Lattice partitions based on signal codes
- An efficient approach to transmit side information for multiple users in a way that is compatible with lattice network coding
Lattice Network Coding
Key concepts

Fine lattice $\Lambda$, coarse lattice $\Lambda' \subseteq \Lambda$, and lattice partition $\Lambda/\Lambda'$

$$G_\Lambda = \begin{bmatrix} \sqrt{3} & 1 \\ 0 & 2 \end{bmatrix}$$

$$\Lambda = \{ rG_\Lambda : r \in \mathbb{Z}^2 \}$$

$$\Lambda' = 3\Lambda$$
Key concepts

Message space $W$ (with $|W| = |\Lambda/\Lambda'|$)

Labeling $\phi : \Lambda \rightarrow W$ (consistent with $\Lambda/\Lambda'$)

Embedding map $\bar{\phi} : W \rightarrow \Lambda$ such that $\phi(\bar{\phi}(w)) = w$

$Lattice Network Coding$

$W = \mathbb{Z}_3 \times \mathbb{Z}_3$

$\phi(wG_\Lambda) = w \mod 3$

$\bar{\phi}(w) = wG_\Lambda$
Natural projection

For any discrete subring $R \subseteq \mathbb{C}$ such that $\Lambda / \Lambda'$ is an $R$-module, we can make $W$ into an $R$-module such that $\varphi : \Lambda \to W$ is a surjective $R$-module homomorphism with kernel $\Lambda'$.

\[
R = \mathbb{Z}
\]

\[
W = \mathbb{Z}_3 \times \mathbb{Z}_3
\]

\[
\varphi(\bar{\varphi}(2, 1) + \bar{\varphi}(1, 0)) = (0, 1)
\]
Transmitter $\ell$ sends $x_\ell = \overline{\varphi}(w_\ell) + \lambda'_\ell$

Receiver computes $\hat{u} = \varphi(\mathcal{D}_\Lambda(\alpha y))$

Error Probability

$$\Pr[\text{error}] = \Pr[\mathcal{D}_\Lambda(n_{\text{eff}}) \notin \Lambda']$$

where $n_{\text{eff}} \triangleq \sum_\ell (\alpha h_\ell - a_\ell) x_\ell + \alpha z$ is the effective noise.

Remark: hypercube shaping $\implies$ UBE based on lattice parameters
Signal Codes
Generator matrix

\[
G_{\Lambda}^{k \times (k+m)} = \begin{bmatrix}
1 & g_1 & \cdots & g_m & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & g_{m-1} & g_m & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & g_{m-1} & g_m \\
\end{bmatrix}
\]

where \( g_i \in \mathbb{C} \), for \( i = 1, \ldots, m \).

Encoding and decoding operations

- **Encoding**: convolution + shaping + termination
  \[
x = w G_{\Lambda} + \chi^\prime + d, \text{ where } G_{\Lambda'} = \pi G_{\Lambda}
\]
- **Decoding**: sequential decoding
Convolution: $wG_{\Lambda}$, or equivalently, $(w_1, \ldots, w_k) \ast (1, g_1, \ldots, g_m)$
Convolution: $w G_{\Lambda}$, or equivalently, $(w_1, \ldots, w_k) \ast (1, g_1, \ldots, g_m)$
Encoding

Shaping: $w\mathbf{G}_\Lambda + \lambda'$, where $\lambda' = w'\mathbf{G}_\Lambda \prime = w'\pi\mathbf{G}_\Lambda$ and

$$w'_i = -\left( w_i + \sum_{j=1}^m g_j (w_{i-j} + \pi w'_{i-j}) \right)/\pi $$
Shaping: $wG_{\Lambda'} + \chi'$, where 

$$\chi' = w'G_{\Lambda'} = w'\pi G_{\Lambda}$$

and 

$$w'_i = - \left\lceil \frac{(w_i + \sum_{j=1}^{m} g_j (w_{i-j} + \pi w'_{i-j}))}{\pi} \right\rceil$$

Add $-3D^2g(D)$
Shaping: \( wG_\Lambda + \chi' \), where \( \chi' = w'G_\Lambda' = w'\pi G_\Lambda \) and

\[
w'_i = - \left( w_i + \sum_{j=1}^{m} g_j (w_{i-j} + \pi w'_{i-j}) \right) / \pi \]
Shaping: $w G_{\Lambda} + \lambda'$, where $\lambda' = w' G_{\Lambda'} = w' \pi G_{\Lambda}$ and

$$w'_i = - \left[ (w_i + \sum_{j=1}^{m} g_j (w_{i-j} + \pi w'_{i-j})) / \pi \right]$$
Encoding

Shaping: $wG_\Lambda + \chi'$, where $\chi' = w'G_{\Lambda'} = w'\pi G_\Lambda$ and

$$w'_i = -\left[\frac{(w_i + \sum_{j=1}^{m} g_j (w_{i-j} + \pi w'_{i-j}))}{\pi}\right]$$
Termination: \( w G_\Lambda + \lambda' + d \), where \( d \) is a function of \( w G_\Lambda + \lambda' \)
Termination: $wG_\Lambda + \lambda' + d$, where $d$ is a function of $wG_\Lambda + \lambda'$
Lattice Network Coding via Signal Codes
Encoder
Transmitter $\ell$ sends $x_\ell = w_\ell G_\Lambda + \lambda'_\ell + d_\ell$, where $\lambda'_\ell = w'_\ell \pi G_\Lambda$
and $d_\ell$ is a function of $w_\ell G_\Lambda + \lambda'_\ell$

Decoder
Receiver computes $D_\Lambda (\alpha y - \sum_{\ell} a_\ell d_\ell)$

Error Probability
$\Pr[\text{error}] = \Pr[D_\Lambda (n_{\text{eff}}) \not\in \Lambda']$, where the effective noise $n_{\text{eff}}$ is
$n_{\text{eff}} \triangleq \sum_{\ell} (\alpha h_\ell - a_\ell) x_\ell + \alpha z$

Achieving coding gains
From (CISS 2011), coding gains of signal codes are achievable as long as side information $\sum_{\ell} a_\ell d_\ell$ is available at the receiver
## Question

How shall transmitters send their side information $d_1, \ldots, d_L$?

## Possible solutions

- use side channels
- use a central coordinator

## Our solution

use the idea behind compute-and-forward

- transmitter $\ell$ sends $f(d_\ell)$ concurrently
- receiver only computes $\sum_\ell a_\ell d_\ell$
Recall that in our solution...

Transmitter $\ell$ sends $f(d_\ell)$, where $d_\ell \in \mathbb{Z}[i]^m$.

Receiver computes $\sum_\ell a_\ell d_\ell$.

Recall that in lattice network coding...

Transmitter $\ell$ sends $x_\ell = \mathcal{E}(w_\ell)$, where $W = (\mathbb{Z}[i]/(q))^m$.

Receiver computes $\sum_\ell a_\ell w_\ell$.

Key observation

If $q$ is large enough, then $W$ is as good as $\mathbb{Z}[i]^m$ for $d_\ell$, and the problem reduces to a lattice design problem.
A mathematical formulation for our design problem

Let \( \Lambda_q / \Lambda'_q \) be a lattice partition with message space \( W \)

\[
\begin{align*}
\text{minimize} & \quad \text{dimension of } \Lambda_q \\
\text{subject to} & \quad d^2(\Lambda_q / \Lambda'_q) \geq \gamma \\
& \quad |q| \geq 2\theta
\end{align*}
\]

dimension of \( \Lambda_q \) corresponds to the extra channel uses;
\( \gamma \) captures the reliability of decoding \( \sum_{\ell} a_{\ell} d_{\ell} \)
\( \theta \) captures the dynamic range of \( \sum_{\ell} a_{\ell} d_{\ell} / \pi \)
Recall the design problem

\[
\begin{align*}
\text{minimize} & \quad \text{dimension of } \Lambda_q \\
\text{subject to} & \quad d^2(\Lambda_q/\Lambda'_q) \geq \gamma \\
& \quad |q| \geq 2\theta
\end{align*}
\]

We give a constructive method showing that

\[
\text{dimension of } \Lambda_q \leq m \sqrt{1.57 \gamma (|p_1| + |p_2| + \ldots + |p_s|)},
\]

where \( p_i \) are Gaussian primes, and \(|p_1p_2\cdots p_s| \geq 2\theta\)
A concrete example

For signal codes $\Lambda$ and $\Lambda'$

- dimension: 2000
- parameters: $\{g_1 = 1.96e^{i\pi/8}, g_2 = 0.98^2e^{i\pi/4}\}$
- $\Lambda' = 3\Lambda$
- coding gain: 6.4 dB (over uncoded 9-QAM)

For the design problem

- $\gamma = 1$: corresponds to 9.5 dB protection
- $\theta = 20000$: covers the dynamic range for two users

For our constructive method

- analysis: 90 extra channel uses
- simulation: 92 extra channel uses

Thus, overhead $\approx 5\%$
Simulation Results

Setup

- two-transmitter, single-receiver multiple-access
- Rayleigh faded channel gains
- receiver decodes $a_1 w_1 + a_2 w_2$ ($a_1, a_2 \neq 0$)
- $9$-QAM LNC: $\Lambda / \Lambda' = \mathbb{Z}[i]^{2000} / 3\mathbb{Z}[i]^{2000}$
- $9$-QAM PNC: Zhang, Liew & Lam; Popovski & Yomo
Simulation Results

Frame-Error Rate vs SNR [dB]

- Nazer-Gastpar
- Signal-Code
- 9-QAM LNC
- 9-QAM PNC

Lattice Network Coding via Signal Codes
Conclusions

1. **Potential Performance Gain**
   - coding gains of signal codes can be achieved
   - but side information is required

2. **Methods for Sending Side Information**
   - a generic scheme
   - a lattice design problem

3. **Overhead of Sending Side Information**
   - an upper bound
   - a concrete example
Thank You!
Recall that...

Transmitter sends $x = w_G + \lambda' + d$, where $\lambda' = w'\pi G_\Lambda$ and $d$ is a function of $w_G + \lambda'$.

Decoding

Receiver computes $D_\Lambda(y - d) = w_G + w'\pi G_\Lambda + D_\Lambda(z)$.

If $D_\Lambda(z) = 0$, then the receiver obtains $w + \pi w'$.

Note that $[w + \pi w'] \mod \pi = w \mod \pi$. 
Advantages of Signal Codes

Coding gains

Recall that a signal code is “generated by” \([1, g_1, \ldots, g_m]\)

- when \(m = 2\), coding gain = 6.4dB
- when \(m = 3\), coding gain = 8.3dB
- when \(m = 4\), coding gain = 9.6dB

Achieving coding gains

coding gains of signal codes are achievable as long as side information is available at the receiver