Design Criteria for Lattice Network Coding

Chen Feng
Department of Electrical & Computer Engineering
University of Toronto, Canada

joint work with:
Danilo Silva, Federal University of Santa Catarina, Brazil, and
Frank R. Kschischang, University of Toronto

CISS 2011, Baltimore, MD, USA
23 March, 2011
Compute-and-Forward Relaying Strategy
Compute-and-Forward Relaying Strategy
Compute-and-Forward Relaying Strategy

\[ y_3 = h_{13} x_1 + h_{23} x_2 + z_3 \]

\[ y_4 = h_{14} x_1 + h_{24} x_2 + z_4 \]
Compute-and-Forward Relaying Strategy

\[ y_3 = h_{13} x_1 + h_{23} x_2 + z_3 \]

\[ w_3 = a_{13} w_1 + a_{23} w_2 \]

\[ y_4 = h_{14} x_1 + h_{24} x_2 + z_4 \]

\[ w_4 = a_{14} w_1 + a_{24} w_2 \]
Compute-and-Forward Relaying Strategy

\[ w_3 = a_{13} w_1 + a_{23} w_2 \]

\[ w_4 = a_{14} w_1 + a_{24} w_2 \]
Compute-and-Forward Relaying Strategy

\[ y_5 = h_{35}x_3 + h_{45}x_4 + z_5 \]

\[ w_3 = a_{13}w_1 + a_{23}w_2 \]

\[ w_4 = a_{14}w_1 + a_{24}w_2 \]
Compute-and-Forward Relaying Strategy

$w_3 = a_{13} w_1 + a_{23} w_2$

$w_4 = a_{14} w_1 + a_{24} w_2$

$w_5 = a_{35} w_3 + a_{45} w_4$

$y_5 = h_{35} x_3 + h_{45} x_4 + z_5$
Problem Setup

Assumption

The receiver is only interested in a linear combination $u$
Problem Setup

\[ \begin{align*}
    w_1 & \in \mathbb{F}_q^k & x_1 & \in \mathbb{C}^n \\
    w_2 & \in \mathbb{F}_q^k & x_2 & \in \mathbb{C}^n \\
    \vdots & \vdots & \vdots & \vdots \\
    w_L & \in \mathbb{F}_q^k & x_L & \in \mathbb{C}^n \\
\end{align*} \]

\[ \begin{align*}
    u &= \sum_{\ell=1}^{L} a_{\ell} w_{\ell} \\
    y &= \sum_{\ell=1}^{L} h_{\ell} x_{\ell} + z
\end{align*} \]

\[ \hat{u} \]

Assumption

Same power constraint \( P \) for all transmitters
**Problem Setup**

The encoder rate $R \triangleq k \log_2 q / n$; probability of error $P_e \triangleq \Pr[\hat{u} \neq u]$. 

$w_1 \in \mathbb{F}_q^k \rightarrow \mathcal{E} \rightarrow x_1 \in \mathbb{C}^n \rightarrow u = \sum_{\ell=1}^L a_{\ell}w_{\ell}$

$w_2 \in \mathbb{F}_q^k \rightarrow \mathcal{E} \rightarrow x_2 \in \mathbb{C}^n \rightarrow \vdots \rightarrow \mathcal{E} \rightarrow x_L \in \mathbb{C}^n \rightarrow y = \sum_{\ell=1}^L h_{\ell}x_{\ell} + z$

$y \rightarrow \mathcal{D} \rightarrow \hat{u}$

Performance metrics

(a_1, \ldots, a_L)

(h_1, \ldots, h_L)
Nazer-Gastpar’s Approach

A lower bound for the encoder rate $R$: Nazer-Gastpar (2008)

$$R \geq R_{\text{comp}} = \log_2 \left( \frac{\text{SNR}}{Q(a)} \right)$$

where $Q(a) = aM a^\dagger$, and the matrix $M$ is

$$M = \text{SNR} I_L - \frac{\text{SNR}^2}{\text{SNR} \left\| h \right\|^2 + 1} h^\dagger h.$$  

Key idea: asymptotically-good lattice partitions (Erez and Zamir) and MMSE estimation
Some High-Level Questions

- **Which lattice partitions can be used?**
  instead of asymptotically-good lattice partitions

- **What are their performances?**
  in terms of the probability of error $P_e$
Some High-Level Questions

- **Which lattice partitions can be used?**
  
  considered in our ISIT 2010 paper  
  also in the work of Hern and Narayanan (2010)  
  and in the work of Ordentlich and Erez (2010)

- **What are their performances?**
  
  considered in this paper
Problem Considered in This Paper

\[ \mathbf{w}_1 \in \mathbb{F}_{q}^k \rightarrow \mathbf{x}_1 \in \mathbb{C}^n \rightarrow \mathbf{u} = \sum_{\ell=1}^{L} a_\ell \mathbf{w}_\ell \]

\[ \mathbf{w}_2 \in \mathbb{F}_{q}^k \rightarrow \mathbf{x}_2 \in \mathbb{C}^n \rightarrow \mathbf{y} = \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + z \]

**Problem**

Let \( \Lambda / \Lambda' \) be a lattice partition. Let \( \mathbf{a} \) be a coefficient vector. Then what is the probability of error \( P_e \triangleq \Pr[\hat{\mathbf{u}} \neq \mathbf{u}] \)?
Main Result

Let $\Lambda/\Lambda'$ be a lattice partition. Let $\mathbf{a}$ be a coefficient vector. Assume that hypercube shaping is used for $\Lambda/\Lambda'$. Then the union bound estimate of the probability of error $P_e$ is

$$P_e \approx N(\Lambda \setminus \Lambda') \exp \left( -\frac{d^2(\Lambda/\Lambda')}{4\sigma^2 Q(\mathbf{a})} \right)$$

for high signal-to-noise ratios, where $Q(\mathbf{a})$ is exactly the same as that in Nazer-Gastpar’s result.

Definitions of the parameters $N(\Lambda \setminus \Lambda')$ and $d(\Lambda/\Lambda')$

$N(\Lambda \setminus \Lambda')$: number of the shortest vectors in the set difference $\Lambda \setminus \Lambda'$

$d(\Lambda/\Lambda')$: length of the shortest vectors in the set difference $\Lambda \setminus \Lambda'$
Implications of the Main Result

Solve a “disign-time” problem

Question: How shall we choose the lattice partition $\Lambda/\Lambda'$?
Answer: maximize $d(\Lambda/\Lambda')$ and minimize $N(\Lambda \setminus \Lambda')$

Solve a “run-time” problem

Question: How shall we choose the coefficient vector $\mathbf{a}$?
Answer: minimize $Q(\mathbf{a})$
Implications of the Main Result

Solve a “design-time” problem
Question: How shall we choose the lattice partition $\Lambda/\Lambda'$?
Answer: maximize $d(\Lambda/\Lambda')$ and minimize $\mathcal{N}(\Lambda \setminus \Lambda')$

Solve a “run-time” problem
Question: How shall we choose the coefficient vector $\mathbf{a}$?
Answer: minimize $Q(\mathbf{a})$

Remark
Note that these two problems are separable under our assumptions
Proof of the Main Result (real-valued case)

Each transmitter applies the same nested lattice code \( \Lambda/\Lambda' \)
There is a one-to-one linear map between messages and cosets
Proof of the Main Result (real-valued case)

Transmitter 1 maps $w_1$ to a coset representative
Proof of the Main Result (real-valued case)

Transmitter 2 maps $w_2$ to a coset representative
The channel is given by $y = 1.4x_1 + 0.6x_2 + z$
Proof of the Main Result (real-valued case)

\[ y = h_1 x_1 + h_2 x_2 + z \]

\((h_1, h_2) = (1.4, 0.6)\)

Hence, the received signal \(y\) is like this.
Proof of the Main Result (real-valued case)

\[ y = h_1 x_1 + h_2 x_2 + z \]

\[ (h_1, h_2) = (1.4, 0.6) \]

\[ (a_1, a_2) = (2, 1) \]

Receiver picks up a coefficient vector \((a_1, a_2) = (2, 1)\)
Proof of the Main Result (real-valued case)

\[ y = h_1 x_1 + h_2 x_2 + z \]
\[ (h_1, h_2) = (1.4, 0.6) \]
\[ (a_1, a_2) = (2, 1) \]

Receiver scales the received signal to \( \alpha y = 2.04x_1 + 0.88x_2 + 1.46z \)
Proof of the Main Result (real-valued case)

\[ y = h_1 x_1 + h_2 x_2 + z \]

\[ (h_1, h_2) = (1.4, 0.6) \]

\[ (a_1, a_2) = (2, 1) \]

Hence, 
\[ \alpha y = 2x_1 + x_2 + 0.04x_1 - 0.12x_2 + 1.46z = 2x_1 + x_2 + n_{\text{eff}} \]
Proof of the Main Result (real-valued case)

\[ y = h_1 x_1 + h_2 x_2 + z \]
\[ (h_1, h_2) = (1.4, 0.6) \]
\[ (a_1, a_2) = (2, 1) \]

Since the effective noise is small, the decoding is correct.
Proof of the Main Result (real-valued case)

Recall that...

there is a one-to-one linear map between $\mathbb{F}_3^2$ and $\Lambda / \Lambda'$

Finally, map the decoded lattice point to a corresponding message

$w_1 = (1, 0)$

$w_2 = (0, 1)$

$\hat{u} = (2, 1)$

$\hat{u} = 2w_1 + w_2$
Proof of the Main Result (real-valued case)

\( y = h_1 x_1 + h_2 x_2 + z \)

\((h_1, h_2) = (1.4, 0.6)\)

\((a_1, a_2) = (2, 1)\)

\( P_e = \Pr[ n_{\text{eff}} \notin \text{the Voronoi region of } \Lambda] \)

It might seem that \( P_e \)
Proof of the Main Result (real-valued case)

\[ \begin{align*}
  y &= h_1 x_1 + h_2 x_2 + z \\
  (h_1, h_2) &= (1.4, 0.6) \\
  (a_1, a_2) &= (2, 1)
\end{align*} \]

In fact, \( P_e < \Pr[\mathbf{n}_{\text{eff}} \notin \text{the Voronoi region of } \Lambda] \)
The correct answer is $P_e = \Pr[\mathcal{D}_{\Lambda}(n_{\text{eff}}) \notin \Lambda']$
Proof of the Main Result (real-valued case)

\[ y = h_1 x_1 + h_2 x_2 + z \]
\[ (h_1, h_2) = (1.4, 0.6) \]
\[ (a_1, a_2) = (2, 1) \]

Hence, we shall approximate \( \Pr[\mathcal{D}_\Lambda(n_{\text{eff}}) \notin \Lambda'] \)
Proof of the Main Result (real-valued case)

The original problem

Approximate $\Pr[D_{\Lambda}(n_{\text{eff}}) \notin \Lambda']$
Proof of the Main Result (real-valued case)

<table>
<thead>
<tr>
<th>The original problem</th>
<th>Approximate Pr[$\mathcal{D}<em>\Lambda(n</em>{\text{eff}}) \notin \Lambda'$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A simpler problem</td>
<td>Approximate Pr[$\mathcal{D}<em>\Lambda(n</em>{\text{eff}}) \notin \mathbf{0}$]</td>
</tr>
<tr>
<td>The original problem</td>
<td>Approximate $\Pr[\mathcal{D}<em>\Lambda(n</em>{\text{eff}}) \notin \Lambda']$</td>
</tr>
<tr>
<td>----------------------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>A simpler problem</td>
<td>Approximate $\Pr[\mathcal{D}<em>\Lambda(n</em>{\text{eff}}) \notin 0]$</td>
</tr>
<tr>
<td>A much simpler problem</td>
<td>Approximate $\Pr[\mathcal{D}_\Lambda(z) \notin 0]$</td>
</tr>
</tbody>
</table>
Proof of the Main Result (real-valued case)

<table>
<thead>
<tr>
<th>A much simpler problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question: Approximate $\Pr[D_{\Lambda}(z) \not\in 0]$</td>
</tr>
<tr>
<td>Solution: Union bound</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A simpler problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question: Approximate $\Pr[D_{\Lambda}(n_{\text{eff}}) \not\in 0]$</td>
</tr>
<tr>
<td>Solution: Union bound + Chernoff bound</td>
</tr>
</tbody>
</table>
Proof of the Main Result (real-valued case)

A much simpler problem

| Question: Approximate $\Pr[\mathcal{D}_\Lambda(z) \notin 0]$ |
| Solution: Union bound |

A simpler problem

| Question: Approximate $\Pr[\mathcal{D}_\Lambda(n_{\text{eff}}) \notin 0]$ |
| Solution: Union bound + Chernoff bound |

The original problem

| Question: Approximate $\Pr[\mathcal{D}_\Lambda(n_{\text{eff}}) \notin \Lambda']$ |
| Solution: Union bound + Chernoff bound + A Lemma |

Lemma

Consider the set difference $\Lambda \setminus \Lambda'$. Let $R_V(0)$ be the Voronoi region of $0$ in the set $\Lambda \setminus \Lambda' \cup \{0\}$. Then $\Pr[\mathcal{D}_\Lambda(n) \notin \Lambda'] \leq \Pr[n \notin R_V(0)]$
Recall that
\[ \Pr[\mathcal{D}_{\Lambda}(\mathbf{n}) \notin \Lambda'] \leq \Pr[\mathbf{n} \notin \mathcal{R}_V(0)] \]

Finally, we have
Let \( \Lambda/\Lambda' \) be a lattice partition. Let \( \mathbf{a} \) be a coefficient vector. Assume that hypercube shaping is used for \( \Lambda/\Lambda' \). Then the union bound estimate of the probability of error \( P_e \) is

\[
P_e \approx \mathcal{N}(\Lambda \setminus \Lambda') \exp \left( -\frac{d^2(\Lambda/\Lambda')} {4\sigma^2 Q(\mathbf{a})} \right)
\]

for high signal-to-noise ratios, where \( Q(\mathbf{a}) \) is exactly the same as that in Nazer-Gastpar’s result.
Conclusions

Nazer-Gastpar’s result (asymptotically-good lattice partitions)
\[ R \geq R_{\text{comp}} = \log_2 \left( \frac{\text{SNR}}{Q(a)} \right) \]

Main result (lattice partitions that admit hypercube shaping)
\[ P_e \approx N(\Lambda \setminus \Lambda') \exp \left( -\frac{d^2(\Lambda/\Lambda')}{4\sigma^2 Q(a)} \right) \]

Implications of the main result
- The lattice partition \( \Lambda/\Lambda' \) should be chosen such that \( d(\Lambda/\Lambda') \) is maximized and \( N(\Lambda \setminus \Lambda') \) is minimized.
- The coefficient vector \( a \) should be chosen such that \( Q(a) \) is minimized.
- These two problems are separable under our assumptions.
Some Open Problems

- Relax the assumption of hypercube shaping
- Construct good lattice partitions
Thank You!
A Short Summary of Nazer-Gastpar’s Architecture

First, pick up a coefficient vector \((a_1, \ldots, a_L)\)

Then, apply a linear MMSE estimator \(g(y) = \alpha y\)

\[
\alpha y = \sum_{\ell=1}^{L} \alpha h_{\ell} x_{\ell} + \alpha z
\]

\[
= \sum_{\ell=1}^{L} a_{\ell} x_{\ell} + \sum_{\ell=1}^{L} (\alpha h_{\ell} - a_{\ell}) x_{\ell} + \alpha z
\]

\[
= \sum_{\ell=1}^{L} a_{\ell} x_{\ell} + n_{\text{eff}},
\]

where \(a_1, \ldots, a_L \in \mathbb{Z}[i]\) and \(\alpha \in \mathbb{C}\) is the MMSE coefficient.

Finally, map the decoded lattice point \(\sum_{\ell=1}^{L} a_{\ell} x_{\ell}\) to \(\sum_{\ell=1}^{L} a_{\ell} w_{\ell}\).
Design Criterion 1

The coefficient vector \((a_1, \ldots, a_L)\) should be chosen such that 
\[ Q(a) = a^* M a \] is minimized.
Design Criterion 1

The coefficient vector \((a_1, \ldots, a_L)\) should be chosen such that
\[ Q(a) = a^* Ma^\dagger \] is minimized.

It is a shortest vector problem. Why?

Since \(M\) is Hermitian and positive-definite, we have \(M = LL^\dagger\). Hence,
\[ Q(a) = a^* Ma^\dagger = \|aL\|^2 \]
Design Criterion 2

The nested lattice code $\Lambda/\Lambda'$ should be chosen such that $d(\Lambda/\Lambda')$ is maximized.
Applications of Our Design Criteria 2

**Design Criterion 2**

The nested lattice code $\Lambda/\Lambda'$ should be chosen such that $d(\Lambda/\Lambda')$ is maximized.

**It is related to some well-studied problems. Why?**

1. For Construction A, $d^2(\Lambda/\Lambda')$ is proportional to $w_E^{\text{min}}(C)$
2. $w_E^{\text{min}}(C)$ is the minimum Euclidean weight of the linear code $C$
3. $w_E^{\text{min}}(C)$ has the following lower bounds
   - $w_E^{\text{min}}(C) \geq w_H^{\text{min}}(C)$ and $w_E^{\text{min}}(C) \geq w_M^{\text{min}}(C)$, where
     - $w_H^{\text{min}}(C)$: the minimum Hamming weight of $C$
     - $w_M^{\text{min}}(C)$: the minimum Mannheim weight of $C$
4. Hence, we can maximize these lower bounds
Some Comparisons

Recall the results for AWGN channels (high SNR regime)

Shannon: \( R = \log_2(1 + \text{SNR}) \approx \log_2(\text{SNR}) \)

Union bound estimate: \( P_e \approx K(\Lambda) \exp \left( - \frac{d^2(\Lambda)}{4\sigma^2} \right) \)

What are the parameters \( N(\Lambda \setminus \Lambda') \) and \( d(\Lambda / \Lambda') \) then?

\( N(\Lambda \setminus \Lambda') \): number of the shortest vectors in \( \Lambda \setminus \Lambda' \)

\( d(\Lambda / \Lambda') \): length of the shortest vectors in \( \Lambda \setminus \Lambda' \)
Some Comparisons

Recall the results for AWGN channels (high SNR regime)

Shannon: \( R = \log_2(1 + \text{SNR}) \approx \log_2(\text{SNR}) \)

Union bound estimate: \( P_e \approx K(\Lambda) \exp \left( -\frac{d^2(\Lambda)}{4\sigma^2} \right) \)

Compare the results for our problem (high SNR regime)

Nazer-Gastpar: \( R_{\text{comp}} = \log_2 \left( \frac{\text{SNR}}{Q(a)} \right) \)

Union bound estimate: \( P_e \approx \mathcal{N}(\Lambda \setminus \Lambda') \exp \left( -\frac{d^2(\Lambda/\Lambda')}{4\sigma^2 Q(a)} \right) \)

What are the parameters \( \mathcal{N}(\Lambda \setminus \Lambda') \) and \( d(\Lambda/\Lambda') \) then?

- \( \mathcal{N}(\Lambda \setminus \Lambda') \): number of the shortest vectors in \( \Lambda \setminus \Lambda' \)
- \( d(\Lambda/\Lambda') \): length of the shortest vectors in \( \Lambda \setminus \Lambda' \)
Some Comparisons

Recall the results for AWGN channels (high SNR regime)

Shannon:  \( R = \log_2(1 + \text{SNR}) \approx \log_2(\text{SNR}) \)

Union bound estimate:  \( P_e \approx K(\Lambda) \exp \left( -\frac{d^2(\Lambda)}{4\sigma^2} \right) \)

Compare the results for our problem (high SNR regime)

Nazer-Gastpar:  \( R_{\text{comp}} = \log_2 \left( \frac{\text{SNR}}{Q(\mathbf{a})} \right) \)

Union bound estimate:  \( P_e \approx \mathcal{N}(\Lambda \setminus \Lambda') \exp \left( -\frac{d^2(\Lambda/\Lambda')}{4\sigma^2 Q(\mathbf{a})} \right) \)

What are the parameters \( \mathcal{N}(\Lambda \setminus \Lambda') \) and \( d(\Lambda/\Lambda') \) then?

\( \mathcal{N}(\Lambda \setminus \Lambda') \): number of the shortest vectors in \( \Lambda \setminus \Lambda' \)

\( d(\Lambda/\Lambda') \): length of the shortest vectors in \( \Lambda \setminus \Lambda' \)