
I. PROOF OF PROPOSITION 1

Proposition 1. Depending on the 4G coverage $Q_t$, the operator’s revenue in time stage $t + 1$ is:

- **Low 4G coverage regime.** When $Q_t < \Delta P/\Delta R$, no users choose 4G service and users with $\alpha \in [\alpha_{(3,0)}, 1]$ choose 3G. The operator’s current revenue is:

$$R(Q_t) = \rho P^{3G}(1 - P^{3G}/R^{3G}), \quad (1)$$

- **Medium 4G coverage regime.** When $\Delta P/\Delta R \leq Q_t < \Delta PR^{3G}/(P^{3G}\Delta R)$, users with $\alpha \in [\alpha_{(3,0)}, \alpha_{(4,3)}]$ choose 3G, and $\alpha \in [\alpha_{(4,3)}, 1]$ choose 4G. The operator’s current revenue is:

$$R(Q_t) = P^{4G}\left(1 - \frac{P^{4G}}{Q_t\Delta R + R^{3G}}\right), \quad (2)$$

- **High 4G coverage regime.** When $Q_t \geq \Delta PR^{3G}/(P^{3G}\Delta R)$, no users choose 3G service and users with $\alpha \in [\alpha_{(4,0)}, 1]$ choose 4G. The operator’s current revenue is:

$$R(Q_t) = \frac{P^{4G}}{Q_t\Delta R + R^{3G}}, \quad (3)$$

**Proof.** Let $D^{3G}(Q_t)$ and $D^{4G}(Q_t)$ denote the proportion of users who choose 3G or 4G service respectively. There are three cases of users’ service partition as shown in Fig. 1.

- **Low 4G coverage regime**, as shown in Fig. 1(a). When $Q_t < \Delta P/\Delta R$, we can derive that

$$D^{4G}(Q_t) = 0$$

$$D^{3G}(Q_t) = 1 - \alpha_{(3,0)}(Q_t) = \frac{R^{3G} - P^{3G}}{R^{3G}}$$

- **Medium 4G coverage regime**, as shown in Fig. 1(b). When $\Delta P/\Delta R \leq Q_t < \Delta PR^{3G}/(P^{3G}\Delta R)$, we can derive that

$$D^{3G}(Q_t) = \alpha_{(4,3)}(Q_t) - \alpha_{(3,0)}(Q_t)$$

$$= \frac{\Delta PR^{3G} - Q_t\Delta R P^{3G}}{Q_t\Delta RR^{3G}}$$

$$D^{4G}(Q_t) = 1 - \alpha_{(4,3)}(Q_t) = \frac{Q_t\Delta R - \Delta P}{Q_t\Delta R}$$
Fig. 1: Users’ service partitions according to 4G coverages.

- **High 4G coverage regime**, as shown in Fig. 1(c). When \( \Delta PR^{3G}/(P^{3G}\Delta R) \leq Q_t \leq 1 \), we can derive that

\[
D^{3G}(Q_t) = \begin{cases} 
1 - \frac{P^{3G}}{\Delta R}, & \text{if } Q_t < \Delta P \Delta R \\
\frac{\Delta P}{\Delta R} - \frac{P^{3G}}{\Delta R}, & \text{if } \frac{\Delta P}{\Delta R} \leq Q_t < \frac{\Delta PR^{3G}}{P^{3G}\Delta R} \\
0, & \text{if } \frac{\Delta PR^{3G}}{P^{3G}\Delta R} \leq Q_t \leq 1 
\end{cases}
\]

In summary,

\[
D^{3G}(Q_t) = \begin{cases} 
0, & \text{if } Q_t < \Delta P \Delta R \\
\frac{\Delta P}{\Delta R} - \frac{P^{3G}}{\Delta R}, & \text{if } \frac{\Delta P}{\Delta R} \leq Q_t < \frac{\Delta PR^{3G}}{P^{3G}\Delta R} \\
1 - \frac{P^{4G}}{Q_tR^{4G} + (1 - Q_t)R^{3G}}, & \text{if } \frac{\Delta PR^{3G}}{P^{3G}\Delta R} \leq Q_t \leq 1 
\end{cases}
\]

Given that the 4G coverage is \( Q_t \), the total revenue from the two services is

\[
R(Q_t) = P^{4G} \rho D^{4G}(Q_t) + P^{3G} \rho D^{3G}(Q_t)
\]
\[ R(Q_t) = \begin{cases} 
\rho P^{3G}(1 - \frac{P^{3G}}{R^{3G}}), & \text{if } Q_t \leq \frac{\Delta P}{\Delta R} \\
\rho(P^{4G}) - \frac{\Delta t^2}{Q_t^2 R^{3G}} - \frac{(P^{3G})^2}{R^{3G}}, & \text{if } \frac{\Delta P}{\Delta R} \leq Q_t < \frac{\Delta P R^{3G}}{P^{3G} R^{3G}} \\
\rho(P^{4G}) - \frac{(P^{4G})^2}{Q_t^2 R^{3G} + R^{2G}}, & \text{if } \frac{\Delta P R^{3G}}{P^{3G} R^{3G}} \leq Q_t \leq 1 
\end{cases} \]

II. PROOF OF LEMMA 1

**Lemma 1.** \( R(Q_t) \) is concave within two separate \( Q_t \) ranges \([0, \Delta P/\Delta R]\) and \((\Delta P/\Delta R, 1] \), respectively\(^1\), but is just quasi-concave (not concave) in the entire range \( Q_t \in [0, 1] \).

**Proof.** The first derivative of \( R(Q) \) with regard to \( Q \) is (Here we use \( Q \) instead of \( Q_t \) for clarity of expression):

\[ R'(Q) = \begin{cases} 
0, & \text{if } Q \leq \frac{\Delta P}{\Delta R} \\
\frac{\rho \Delta P^2}{Q^2 R^{3G}}, & \text{if } \frac{\Delta P}{\Delta R} \leq Q < \frac{\Delta P R^{3G}}{P^{3G} R^{3G}} \\
\frac{\rho \Delta R (P^{4G})^2}{(Q^2 R^{3G} + R^{2G})^2}, & \text{if } \frac{\Delta P R^{3G}}{P^{3G} R^{3G}} \leq Q \leq 1 
\end{cases} \]

Note that \( R'(Q) \) is continuous at \( Q = \frac{\Delta P R^{3G}}{P^{3G} R^{3G}} \).

The second derivative of \( R(Q) \) with regard to \( Q \) is:

\[ R''(Q) = \begin{cases} 
0, & \text{if } Q \leq \frac{\Delta P}{\Delta R} \\
-2\frac{R^2 P^{4G}^2}{Q^2 R^{3G} + R^{2G}}, & \text{if } \frac{\Delta P}{\Delta R} \leq Q < \frac{\Delta P R^{3G}}{P^{3G} R^{3G}} \\
-2\frac{R^2 P^{4G}^2}{(Q^2 R^{3G} + R^{2G})^2}, & \text{if } \frac{\Delta P R^{3G}}{P^{3G} R^{3G}} \leq Q \leq 1 
\end{cases} \]

\( R'(Q) \) is decreasing in the range \([\Delta P/\Delta R, \Delta P R^{3G}/(P^{3G} \Delta R)] \), and range \([\Delta P R^{3G}/(P^{3G} \Delta R), 1] \), and \( R'(Q) \) is continuous at \( Q = \Delta P R^{3G}/(P^{3G} \Delta R) \). Therefore, \( R'(Q) \) is decreasing in the range \([\Delta P/\Delta R, 1] \), and \( R(Q) \) is concave in the same range.

However, \( R'(Q) \) is 0 in the range \([0, \Delta P/\Delta R] \), but is positive in the range \([\Delta P/\Delta R, \Delta P R^{3G}/(P^{3G} \Delta R)] \). Therefore, \( R(Q) \) is not concave in the entire range \([0, 1] \). \( R(Q) \) is quasi-concave, because \( R(Q) \) is monotonic increasing function, i.e., \( f(\lambda Q_1 + (1 - \lambda)Q_2) \leq \max(f(Q_1), f(Q_2)), \forall \lambda \in [0, 1] \) is always true, because \( \lambda Q_1 + (1 - \lambda)Q_2 \leq \max(Q_1, Q_2), \forall \lambda \in [0, 1] \).

\(^1\)Recall that \( \Delta P/\Delta R < 1 \) as we assume \( P^{3G}/R^{3G} > P^{4G}/R^{4G} \).

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III. PROOF OF PROPOSITION 2

Proposition 2. (Optimal deployment policy before time stage $T_{th}$) It is the best for the operator not to deploy any 4G network before $T_{th}$. That is, $Q^*_t = 0, 1 \leq t \leq T_{th} - 1$.

Proof. First, it is easy to prove that $Q_t = 0, t = 1, \ldots, T_{th} - 1$ is feasible. The objective function is

$$S_0 + \sum_{t=0}^{T_{th}-1} R(Q_t) + \sum_{t=T_{th}}^{T} R(Q_t) - kQ_T$$

$$= S_0 + T_{th} \rho P^{3G} (1 - P^{3G}/R^{3G}) + \sum_{t=T_{th}}^{T} R(Q_t) - kQ_T$$

We can see that the objective function is irrelevant of $Q_t, t = 1, \ldots, T_{th} - 1$. We can also prove that the constraints are irrelevant of $Q_t, t = 1, \ldots, T_{th} - 1$. Therefore, any feasible $Q_t, t = 1, \ldots, T_{th} - 1$ will generate the same objective function value. \hfill \Box

IV. PROOF OF PROPOSITION 3

Proposition 3. (Optimal deployment policy after time stage $T_{th}$) Given any $T_{th}$ value, there exists a mature deployment stage $\bar{T}_{th}$, before and after which the operator has different deployment strategies. The value of $\bar{T}_{th}$ is determined by Algorithm 1. The special case of $\bar{T}_{th} = T_{th} - 1$ leads to no further deployment after $T_{th}$, i.e., $Q^*_t = \Delta P/\Delta R$ for any time $t \in \{T_{th}, T\}$. More generally, when $\bar{T}_{th} < T_{th} - 1$, we have:

- **Aggressive deployment period:** In the time period $t \in [T_{th}, \bar{T}_{th}]$, the operator will use up all his current budget at each time stage $t$ for 4G deployment, i.e., $Q^*_t = Q^*_m$ in (10), the maximum achievable coverage that can be supported by the budget at time stage $t$.

- **Conservative deployment period:** When $t = \bar{T}_{th}$, the operator will conservatively upgrade according to:

$$\begin{align*}
Q^*_{t-1}, & \text{ if } C_d > (T + T_{th} - t) \max\{R'(Q^*_{t-1}), R'(Q^*_m)\} \\
q^*_t, & \text{ if } C_d \in [(T + T_{th} - t)R'(Q^*_{t-1}), (T + T_{th} - t)R'(Q^*_m)] \\
Q^*_m, & \text{ if } C_d < (T + T_{th} - t) \min\{R'(Q^*_{t-1}), R'(Q^*_m)\}
\end{align*}$$

in which $q^*_t$ is the unique solution to the equation $C_d = (T + T_{th} - t)R'(q^*_t)$.

- **No deployment period.** When $\bar{T}_{th} + 1 \leq t \leq T$, $Q^*_t = Q^*_{t-1}$. 

Proof. We first give the solution to the following problem, then use the solution to address the problem in Proposition 3.

\[
\max_{Q_1, \ldots, Q_T} \quad S_0 + C_d Q_0 + \sum_{t=0}^{T} R(Q_t) - C_d Q_T
\]

s.t. \( C_d Q_t \leq S_0 + C_d Q_0 + \sum_{\tau=0}^{t-1} R(Q_\tau), t = 1, \ldots, T \)

\( Q_0 \leq Q_1 \leq Q_2 \leq \cdots \leq Q_T \leq 1 \)

in which \( Q_0 = \Delta P/\Delta R \).

It is easy to see that the problem in Proposition 3 is equivalent to the above optimization problem, with initial coverage \( \Delta P/\Delta R \), initial capital \( (T_{th} - 1)\rho P^{3G}(1 - P^{3G}/R^{3G}) - C_d\Delta P/\Delta R \), and time horizon \( T + 1 - T_{th} \). The above optimization problem is convex optimization, so we can use KKT conditions to derive the optimal solution.

We rewrite the constraints as

\[
Q_t - Q_{t+1} \leq 0, t = 0, \ldots, T - 1
\]

\( Q_T - 1 \leq 0 \)

\[
C_d Q_t - S_0 - C_d Q_0 - \sum_{\tau=0}^{t-1} R(Q_\tau) \leq 0, t = 1, \ldots, T
\]

We define the Lagrangian \( L \) associated with the problem as

\[
L = S_0 + C_d Q_0 + \sum_{t=0}^{T} R(Q_t) - C_d Q_T
\]

\[
- \sum_{t=0}^{T-1} \lambda_t (Q_t - Q_{t+1}) - \lambda_T (Q_T - 1)
\]

\[
- \sum_{t=1}^{T} \mu_t (C_d Q_t - S_0 - C_d Q_0 - \sum_{\tau=0}^{t-1} R(Q_\tau))
\]

The KKT conditions are:

\[
\lambda_t \geq 0, t = 0, \ldots, T
\]

\[
\mu_t \geq 0, t = 1, \ldots, T
\]
\[ Q_t - Q_{t+1} \leq 0, t = 0, \ldots, T - 1 \]
\[ Q_T - 1 \leq 0 \]
\[ C_d Q_t - S_0 - C_d Q_0 - \sum_{\tau=0}^{t-1} R(Q_\tau) \leq 0, t = 1, \ldots, T \]
\[ \lambda_t(Q_t - Q_{t+1}) = 0, t = 0, \ldots, T - 1 \]
\[ \lambda_T(Q_T - 1) = 0 \]
\[ \mu_t(C_d Q_t - S_0 - C_d Q_0 - \sum_{\tau=0}^{t-1} R(Q_\tau)) = 0, t = 1, \ldots, T \]
\[ \frac{\partial L}{\partial Q_t} = (1 + \sum_{\tau=t+1}^{T} \mu_\tau) R'(Q_t) + \lambda_{t-1} - \lambda_t - C_d \mu_t = 0, t \in [1, T - 1] \]
\[ \frac{\partial L}{\partial Q_T} = R'(Q_T) - C_d + \lambda_{T-1} - \lambda_T - C_d \mu_T = 0 \]

We give the solution to the KKT conditions as follows.

There exists a solution \((Q_1^*, \ldots, Q_T^*, \lambda_0, \ldots, \lambda_T, \mu_1, \ldots, \mu_T)\), in which \(Q_t^*, t = 1, \ldots, T\) is given defined in Proposition 3, and \(T_{th}\) in the Algorithm 1.

If \(\bar{T}_{th} = 0\),

\[ \lambda_t = \begin{cases} C_d - (T - t) R'(Q_0^*), & t = 0, \ldots, T - 1 \\ 0, & t = T \end{cases} \]
\[ \mu_t = 0, t = 1, \ldots, T \]

If \(1 \leq \bar{T}_{th} \leq T\),

\[ \lambda_t = \begin{cases} 0, & t = 0, \ldots, \bar{T}_{th} - 1 \\ C_d - (T - t) R'(Q_{\bar{T}_{th}}^*), & t = \bar{T}_{th}, \ldots, T - 1 \\ 0, & t = T \end{cases} \]

\[ \mu_t = \begin{cases} (1 + \sum_{\tau=\min(T, t+1)}^{T} \mu_\tau) R'(Q_t^*) / C_d, & t = 1, \ldots, \bar{T}_{th} - 1 \\ 0, & t = \bar{T}_{th}, \ldots, T \end{cases} \]

Note that if \(\bar{T}_{th} = T, \lambda_t = 0, t = 0, \ldots, T\).

Note that if \(\bar{T}_{th} = 1, \mu_t = 0, t = 1, \ldots, T\).
Lemma 2. The optimal $T_{th}$ is chosen from the following two candidates:

- $T_{th} = T + 1$, that is, the operator will never deploy the 4G network to a coverage level $\Delta P / \Delta R$;
• $T_{th} = [(C_d \Delta P / \Delta R - S_0) / [\rho \mathcal{P}^{3G} (1 - \mathcal{P}^{3G} / \mathcal{R}^{3G})]]$, that is, the operator deploys the 4G network to the coverage level $\Delta P / \Delta R$ as soon as possible.

**Proof.** We only have to prove that if $T_{th} \in [(C_d \Delta P / \Delta R - S_0) / [\rho \mathcal{P}^{3G} (1 - \mathcal{P}^{3G} / \mathcal{R}^{3G})]]$, the optimal one is $T_{th}^* = [(C_d \Delta P / \Delta R - S_0) / [\rho \mathcal{P}^{3G} (1 - \mathcal{P}^{3G} / \mathcal{R}^{3G})]]$.

The original optimization problem is equivalent to

$$
\max_{Q_1, \ldots, Q_{T+1-T_{th}}} (T_{th} - 1) \rho \mathcal{P}^{3G} (1 - \frac{\mathcal{P}^{3G}}{\mathcal{R}^{3G}}) + \sum_{t=0}^{T+1-T_{th}} R(Q_t) - C_d Q_{T+1-T_{th}}
$$

s.t. $C_d Q_t \leq (T_{th} - 1) \rho \mathcal{P}^{3G} (1 - \frac{\mathcal{P}^{3G}}{\mathcal{R}^{3G}}) + \sum_{\tau=0}^{t-1} R(Q_\tau)$, $t = 1, \ldots, T + 1 - T_{th}$

$$Q_0 \leq Q_1 \leq \cdots \leq Q_{T+1-T_{th}} \leq 1$$

in which $Q_0 = \Delta P / \Delta R$ and $R(Q_0) = \rho \mathcal{P}^{3G} (1 - \mathcal{P}^{3G} / \mathcal{R}^{3G})$. Let $OPT(T_{th})$ denote the optimal value of the above objective function. We only have to prove that $OPT(T_{th}) - OPT(T_{th}) \geq 0, T_{th} \leq T - 1$.

We define another optimization problem to help us compare $OPT(T_{th})$ and $OPT(T_{th} + 1)$.

$$
\max_{Q_1, \ldots, Q_{T+1-T_{th}}} \sum_{t=1}^{T+1-T_{th}} R(Q_t) - C_d Q_{T+1-T_{th}}
$$

s.t. $Q_0 \leq Q_1 \leq \cdots \leq Q_{T+1-T_{th}} \leq 1$

$$C_d Q_t \leq (T_{th} - 1) \rho \mathcal{P}^{3G} (1 - \frac{\mathcal{P}^{3G}}{\mathcal{R}^{3G}}) + \sum_{\tau=0}^{t-1} R(Q_\tau)$, $t = 1, \ldots, T + 1 - T_{th}$ (4)

Let $\Phi^*(T_{th})$ denote the optimal value of the above objective function. Therefore, $OPT(T_{th}) - OPT(T_{th} + 1) = \Phi^*(T_{th}) - \Phi^*(T_{th} + 1) - \rho \mathcal{P}^{3G} (1 - \mathcal{P}^{3G} / \mathcal{R}^{3G}), T_{th} \leq T - 1$. Let $Q_t^*(T_{th})$ denote the optimal coverage at time stage $t$ when the time horizon is $T + 1 - T_{th}$.

(1) If $Q_t^*(T_{th}) \leq Q_t^*(T_{th} + 1)$, then $Q_t^*(T_{th}), Q_t^*(T_{th} + 1), Q_t^*(T_{th} + 1), \cdots, Q_t^*(T_{th} + 1)$ is a feasible solution to the problem (4). We have
The special case is that, if the deployment cost satisfies

\[ C_d > \left( T + 1 - \left[ \frac{C_d \Delta P / \Delta R - S_0}{\rho P^{3G}(1 - P^{3G} / R^{3G})} \right] \right) \rho \Delta R, \]  

then 4G network will not be deployed.

VI. PROOF OF THEOREM 1

**Theorem 1.** The optimal 4G deployment policy is one of the following two options.

- **No deployment scheme:** the operator never deploys any 4G network, i.e., \( Q_t^* = 0, t = 1, \ldots, T \).

- **Threshold-based deployment scheme:** Set threshold \( T_{th}^* = \left[ (C_d \Delta P / \Delta R - S_0) / [\rho P^{3G}(1 - P^{3G} / R^{3G})] \right] \).
  - When \( t \in [1, T_{th}^* - 1] \), the operator does not deploy any 4G network, i.e., \( Q_t^* = 0 \);
  - When \( t \in [T_{th}^*, T] \), the operator deploys 4G network according to Proposition 3.
Proof. No deployment scheme corresponds to $T_{th} = T + 1$ in Lemma 2. Threshold-based deployment scheme corresponds to $T_{th} = [(C_d \Delta P/\Delta R - S_0)/[\rho P^{3G}(1 - P^{3G}/R^{3G})]]$ in Lemma 2. The proof of the special case is as follows.

Let $T_1 = [(C_d \Delta P/\Delta R - S_0)/[\rho P^{3G}(1 - P^{3G}/R^{3G})]]$ if $C_d \geq (T+1-T_1)R'(\Delta P/\Delta R)$, then $Q^*_t(T_1) = \Delta P/\Delta R, t = T_{th}, \ldots, T$, then $OPT(T_1) - OPT(T+1) = (T+1-T_1)[R(\Delta P/\Delta R) - \rho P^{3G}(1 - P^{3G}/R^{3G})] - C_d \Delta P/\Delta R = -C_d \Delta P/\Delta < 0$. So we have $T^*_{th} = T + 1$. □

VII. Proof of Proposition 2 when there is operational cost

Proof. First, it is easy to prove that $Q_t = 0, t = 1, \ldots, T_{th} - 1$ is feasible. The objective function is

$$S_0 + \sum_{t=0}^{T_{th}-1} (R(Q_t) - C_o Q_t) + \sum_{t=T_{th}}^{T} (R(Q_t) - C_o Q_t) - C_d Q_T$$

$$= S_0 + T_{th} \rho P^{3G}(1 - P^{3G}/R^{3G}) - \sum_{t=0}^{T_{th}-1} C_o Q_t + \sum_{t=T_{th}}^{T} (R(Q_t) - C_o Q_t) - C_d Q_T$$

We can see that the objective function monotonically decreases with $Q_t, t = 1, \ldots, T_{th} - 1$. And the budget also monotonically decreases with $Q_t, t = 1, \ldots, T_{th} - 1$. Therefore, $Q_t = 0, t = 1, \ldots, T_{th} - 1$ is the optimal solution. □

VIII. Proof of Proposition 3 when there is operational cost

Proof. We first give the solution to the following problem, then use the solution to address the optimization problem when there is operational cost.

$$\max_{Q_1, \ldots, Q_T} \quad S_0 + C_d Q_0 + \sum_{t=0}^{T} (R(Q_t) - C_o Q_t) - C_d Q_T$$

s.t. $Q_t - Q_{t+1} \leq 0, t = 0, \ldots, T - 1$

$Q_T - 1 \leq 0$

$C_d Q_t - S_0 - k Q_0 - \sum_{\tau=0}^{t-1} (R(Q_\tau) - C_o Q_\tau) \leq 0, t = 1, \ldots, T$

in which $Q_0 = \Delta P/\Delta R$.

It is easy to see that the optimization problem in Proposition 3 when there is operational cost is equivalent to the above optimization problem, with initial coverage $\Delta P/\Delta R$, initial capital
\((T_{th} - 1)P^3G (1 - P^3G/R^3G) - (C_d - C_o)\Delta P/\Delta R\), and time horizon \(T + 1 - T_{th}\). The above optimization problem is convex optimization. So we only have to find the solution that satisfies the KKT conditions, which is guaranteed to be optimal.

The Lagrangian \(L\) associated with the problem is

\[
L = S_0 + C_d Q_0 + \sum_{t=0}^{T} R(Q_t) - C_d Q_T - C_o \sum_{t=0}^{T} Q_t - \sum_{t=0}^{T-1} \lambda_t(Q_t - Q_{t+1}) - \lambda_T(Q_T - 1) - \sum_{t=1}^{T} \mu_t[C_d Q_t - S_0 - C_d Q_0 - \sum_{\tau=0}^{t-1} (R(Q_\tau) - C_o Q_\tau)]
\]

The KKT conditions are:

\[
\begin{align*}
\lambda_t &\geq 0, t = 0, \cdots, T \\
\mu_t &\geq 0, t = 1, \cdots, T \\
Q_t - Q_{t+1} &\leq 0, t = 0, \cdots, T - 1 \\
Q_T - 1 &\leq 0 \\
C_d Q_t - S_0 - C_d Q_0 - \sum_{\tau=0}^{t-1} [R(Q_\tau) - C_o Q_\tau] &\leq 0, t = 1, \cdots, T \\
\lambda_t(Q_t - Q_{t+1}) &\geq 0, t = 0, \cdots, T - 1 \\
\lambda_T(Q_T - 1) &\geq 0 \\
\mu_t(C_d Q_t - S_0 - C_d Q_0 - \sum_{\tau=0}^{t-1} (R(Q_\tau) - C_o Q_\tau)) &\geq 0, t = 1, \cdots, T \\
\frac{\partial L}{\partial Q_t} &= (1 + \sum_{\tau=t+1}^{T} \mu_\tau)(R'(Q_t) - C_o) + (\lambda_{t-1} - \lambda_t) - C_d \mu_t = 0, t = 1, \cdots, T - 1 \\
\frac{\partial L}{\partial Q_T} &= R'(Q_T) - C_o - C_d + \lambda_{T-1} - \lambda_T - C_d \mu_T = 0
\end{align*}
\]

There exists a solution \((Q^*_1, \cdots, Q^*_T, \lambda_0, \cdots, \lambda_T, \mu_1, \cdots, \mu_T)\), in which \(Q^*_t, t = 1, \cdots, T\) is the optimal coverage level defined by the adaptation of Proposition 2 when there is operational cost, and
If $\bar{T}_{th} = 0$, 

$$
\lambda_t = \begin{cases} 
C_d - (T - t)(R'(Q_0) - C_o), & t = 0, \ldots, T - 1 \\
0, & t = T 
\end{cases}
$$

$$
\mu_t = 0, t = 1, \ldots, T
$$

If $1 \leq \bar{T}_{th} \leq T$, 

- If $C_d \in [(T + T_{th} - t)(R'(Q_{t-1}^*) - C_o), (T + T_{th} - t)(R'(Q_t^m) - C_o)]$: 

$$
\lambda_t = \begin{cases} 
0, & t = 0, \ldots, \bar{T}_{th} - 1 \\
C_d - (T - t)(R'(Q_{\bar{T}_{th}}^*) - C_o), & t = \bar{T}_{th}, \ldots, T - 1 \\
0, & t = T 
\end{cases}
$$

Note that if $\bar{T}_{th} = T$, $\lambda_t = 0, t = 0, \ldots, T$.

$$
\mu_t = \begin{cases} 
[(1 + \sum_{\tau = \min(T, t+1)}^{T} \mu_{\tau})(R'(Q_{t}^*) - C_o)]/C_d, & t = 1, \ldots, \bar{T}_{th} - 1 \\
0, & t = \bar{T}_{th}, \ldots, T 
\end{cases}
$$

Note that if $\bar{T}_{th} = 1, \mu_t = 0, t = 1, \ldots, T$.

- If $C_d > \max\{(T + T_{th} - t)(R'(Q_{t-1}^*) - C_o), (T + T_{th} - t)(R'(Q_t^m) - C_o)\}$:

$$
\lambda_t = \begin{cases} 
0, & t = 0, \ldots, \bar{T}_{th} - 2 \\
C_d - (T - t)(R'(Q_{\bar{T}_{th} - 1}^*) - C_o), & t = \bar{T}_{th} - 1, \ldots, T - 1 \\
0, & t = T 
\end{cases}
$$

Note that $\bar{T}_{th} \geq 2$ because $C_d - TR'(Q_0) \leq 0$.

$$
\mu_t = \begin{cases} 
[(1 + \sum_{\tau = \min(T, t+1)}^{T} \mu_{\tau})(R'(Q_{t}^*) - C_o)]/C_d, & t = 1, \ldots, \bar{T}_{th} - 2 \\
(T - (\bar{T}_{th} - 2))(R'(Q_{\bar{T}_{th} - 1}^*) - C_o)/C_d - 1, & t = \bar{T}_{th} - 1 \\
0, & t = \bar{T}_{th}, \ldots, T 
\end{cases}
$$

Note that if $\bar{T}_{th} = 2, \mu_1 = T(R'(Q_1^*) - C_o)/C_d - 1, \mu_t = 0, t = 2, \ldots, T$.

- If $C_d < \min\{(T + T_{th} - t)(R'(Q_{t-1}^*) - C_o), (T + T_{th} - t)(R'(Q_t^m) - C_o)\}$, but $Q_t^m = 1$:

$$
\lambda_t = \begin{cases} 
0, & t = 0, \ldots, \bar{T}_{th} - 1 \\
(t + 1 - \bar{T}_{th})(R'(1) - C_o), & t = \bar{T}_{th}, \ldots, T - 1 \\
(T - (\bar{T}_{th} - 1))(R'(1) - C_o) - C_d, & t = T 
\end{cases}
$$
Note that if $\bar{T}_{th} = T, \lambda_t = 0, t = 0, \cdots, T - 1$ and $\lambda_T = R'(1) - C_d$.

$$\mu_t = \begin{cases} 
(1 + \sum_{\tau = \min(T, t+1)}^{T} \mu_{\tau}) (R'(Q^*_t) - C_o) \big/ C_d, & t = 1, \cdots, \bar{T}_{th} - 1 \\
0, & t = T_{th}, \cdots, T 
\end{cases}$$

Note that if $\bar{T}_{th} = 1, \mu_t = 0, t = 1, \cdots, T$.

- $C_d < \min \{(T + T_{th} - t)(R'(Q^*_{t-1}) - C_o), (T + T_{th} - t)(R'(Q^*_t) - C_o)\}$, but $T_{th} = T$: $\lambda_t = 0, t = 0, \cdots, T$

$$\mu_t = \begin{cases} 
(1 + \sum_{\tau = \min(T, t+1)}^{T} \mu_{\tau}) (R'(Q^*_t) - C_o) \big/ C_d, & t = 1, \cdots, T - 1 \\
(R'(Q^*_T) - C_o - C_d) \big/ C_d, & t = T 
\end{cases}$$

Since the solution satisfies the KKT conditions, it is the optimal solution.

\[ \square \]

IX. PROOF OF PROPOSITION 4

**Proposition 4.** Final 4G coverage level $Q^*_T$ increases with time length $T$ and user density $\rho$, but decreases with the deployment cost $C_d$. As $T \to \infty$, $Q^*_T = 1$ (full 4G coverage).

**Proof.** It can be easily proved that $Q^*_T$ increases with $T$ and $\rho$, but decreases with $C_d$. So we only prove that as $T \to \infty$, $Q^*_T = 1$ (full 4G coverage). If the total time horizon $T$ is long enough, $Q^*_t(T_1) = 1, t = \theta, \cdots, T$. Hence $OPT(T_1) - OPT(T + 1) \geq (T + 1 - T_{th} - \theta)(R(1) - \rho P^{3G}(1 - P^{3G}/R^{3G})) - C_d > 0$, because $R(1) - \rho P^{3G}(1 - P^{3G}/R^{3G}) > 0, T \to \infty$. So we have $T_{th} = T_1$.

\[ \square \]

X. PROOF OF LEMMA 3

**Lemma 3.** The optimal $T^*_{th}$ with consideration of the operational cost, is chosen from the following two candidates:

- $T_{th} = T + 1$, that is, the operator will not deploy the 4G coverage to $\Delta P / \Delta R$;
- $T_{th} = \bar{T}$, which satisfies

$$R(Q_{\bar{T}}) - C_o Q_{\bar{T}} \geq R(0) \quad (6)$$

$$R(Q_{\bar{T} - 1}) - C_o Q_{\bar{T} - 1} \leq R(0) \quad (7)$$

**Proof.** We just have to prove that $\bar{T}$ is optimal for $T_{th} \leq T$. 

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Given that the coverage before \( T_{th} \) is zero, the original optimization problem is equivalent to the following optimization problem:

\[
\max_{Q_1, \ldots, Q_{T+1-T_{th}}} (T_{th} - 1) \rho P^{3G} \left(1 - \frac{P^{3G}}{R^{3G}}\right) + \sum_{t=0}^{T+1-T_{th}} (R(Q_t) - C_o Q_t) - C_d Q_{T+1-T_{th}} \\
\text{s.t. } Q_0 \leq Q_1 \leq \cdots \leq Q_{T+1-T_{th}} \leq 1 \\
C_d Q_t \leq (T_{th} - 1) \rho P^{3G} \left(1 - \frac{P^{3G}}{R^{3G}}\right) + \sum_{\tau=0}^{t-1} (R(Q_{\tau}) - C_o Q_{\tau}), \ t = 1, \ldots, T + 1 - T_{th}
\]

in which \( Q_0 = \Delta P / \Delta R \) and \( R(Q_0) = \rho P^{3G} (1 - P^{3G} / R^{3G}) \). Let \( OPT(T_{th}) \) denote the optimal value of the above objective function.

We define another optimization problem to help us compare \( OPT(T_{th}) \) and \( OPT(T_{th} + 1) \).

\[
\max_{Q_1, \ldots, Q_{T+1-T_{th}}} \sum_{t=1}^{T+1-T_{th}} (R(Q_t) - C_o Q_t) - C_d Q_{T+1-T_{th}} \\
\text{s.t. } Q_0 \leq Q_1 \leq \cdots \leq Q_{T+1-T_{th}} \leq 1 \\
C_d Q_t \leq (T_{th} - 1) \rho P^{3G} \left(1 - \frac{P^{3G}}{R^{3G}}\right) + \sum_{\tau=0}^{t-1} (R(Q_{\tau}) - C_o Q_{\tau}), \ t = 1, \ldots, T + 1 - T_{th}
\]

Let \( \Phi^*(T_{th}) \) denote the optimal value of the above objective function. Therefore, \( OPT(T_{th}) - OPT(T_{th} + 1) = \Phi^*(T_{th}) - \Phi^*(T_{th} + 1) - \rho P^{3G} (1 - P^{3G} / R^{3G}), T_{th} \leq T \). Let \( Q_t^*(T_{th}) \) denote the optimal coverage at time stage \( t \), given \( T_{th} \).

We first prove the following argument.

**Argument 1.** For \( T_{th} \leq T \), the value of \( R(\bar{Q}_1^*(T_{th})) - C_o \bar{Q}_1^*(T_{th}) \) has the trend of one of the following two cases:

1) first increases then decreases with \( T_{th} \), or
2) monotonously increases with \( T_{th} \).

**Proof.** When \( T_{th} \) increases, the stopping time \( \bar{T}_{th} \) decreases, because the total time horizon \( T + 1 - T_{th} \) decreases and the upperbound \( Q_1^m \) increases.

- When \( \bar{T}_{th}(T_{th}) > 1 \), \( Q_1^*(T_{th}) = Q_1^m \) increases along with \( T_{th} \). \( R(\bar{Q}_1^*(T_{th})) - C_o \bar{Q}_1^*(T_{th}) \) also increases because \( R'(\bar{Q}_1^*(T_{th})) - C_o > k/(T + 1 - T_{th}) > 0 \).
- When \( \bar{T}_{th}(T_{th}) = 1 \),
In summary, $R(Q_1^*(T_{th})) - C_oQ_1^*(T_{th})$ decreases along with $T_{th}$, $R(Q_1^*(T_{th})) - C_oQ_1^*(T_{th})$ also decreases because $R'(Q_1^*(T_{th})) - C_o = k/(T + 1 - T_{th}) > 0$;

- if $C_d < \max\{ (T + T_{th} - \bar{T}_{th})(R'(Q_1^{th-1}) - C_o), (T + T_{th} - \bar{T}_{th})(R'(Q_m^{th}) - C_o) \}$, $Q_1^*(T_{th})$ remains unchanged.

In summary, $R(Q_1^*(T_{th})) - C_oQ_1^*(T_{th})$ first increases then decreases with $T_{th}$. \hfill \square

Based on Argument 1, we define $T_1 \leq T_2 \leq T$, such that 1) when $T_{th} < T_1$, $R(Q_1^*(T_{th})) - C_oQ_1^*(T_{th}) < \rho P^{3G}(1 - P^{3G}/R^{3G})$; 2) when $T_1 \leq T_{th} < T_2$, $R(Q_1^*(T_{th})) - C_oQ_1^*(T_{th}) \geq \rho P^{3G}(1 - P^{3G}/R^{3G})$; 3) $T_2 \leq T_{th} \leq T$, $R(Q_1^*(T_{th})) - C_oQ_1^*(T_{th}) < \rho P^{3G}(1 - P^{3G}/R^{3G})$. Note that $\bar{T} = T_1$, so we just have to prove that $T_1$ is optimal for $T_{th} \leq T$.

**Argument 2.** When $T_1 \leq T_{th} \leq T_2$, the maximum objective function value is achieved when $T_{th} = T_1$.

**Proof.** We know that $OPT(T_{th}) - OPT(T_{th} + 1) = \Phi^*(T_{th}) - \Phi^*(T_{th} + 1) - \rho P^{3G}(1 - P^{3G}/R^{3G}), T_{th} \in [T_1, T_2 - 1]$. We only have to prove that $OPT(T_{th}) - OPT(T_{th} + 1) \geq 0, T_1 \leq T_{th} \leq T_2$.

1. If $Q_1^*(T_{th}) \leq Q_1^*(T_{th} + 1)$, then $Q_1^*(T_{th}), Q_1^*(T_{th} + 1), Q_2^*(T_{th} + 1), \cdots, Q_{T-T_{th}}^*(T_{th} + 1)$ is a feasible solution to $\Phi(T_{th})$. The proof is as follows. We have
If 

For any $T_{th}$, we know that

$$
\Phi^*(Q_1^*(T_{th}), Q_2^*(T_{th} + 1), \ldots, Q_{T-T_{th}}^*(T_{th} + 1)) \leq \Phi^*(Q_1^*(T_{th}), Q_2^*(T_{th} + 1), \ldots, Q_{T-T_{th}+1}^*(T_{th}))
$$

$$
\Rightarrow \sum_{t=1}^{T-T_{th}} (R(Q_t^*(T_{th} + 1)) - C_oQ_t^*(T_{th} + 1)) - C_dQ_{T-T_{th}}^*(T_{th} + 1) + R(Q_{T-T_{th}}^*(T_{th} + 1)) - C_oQ_{T-T_{th}}^*(T_{th})
$$

$$
= \Phi^*(T_{th} + 1) + R(Q_{T-T_{th}}^*(T_{th} + 1)) - C_oQ_{T-T_{th}}^*(T_{th}) \leq \Phi^*(T_{th})
$$

$$
\Rightarrow \Phi^*(T_{th}) - \Phi^*(T_{th} + 1) \geq R(Q_{T-T_{th}}^*(T_{th} + 1)) - C_oQ_{T-T_{th}}^*(T_{th} + 1) \geq \rho P^3G(1 - P^3G/R^3G)
$$

$$
\Rightarrow OPT(T_{th}) - OPT(T_{th} + 1) \geq 0.
$$

(2) If $Q_1^*(T_{th}) > Q_1^*(T_{th} + 1)$, then $Q_1^*(T_{th} + 1), \ldots, Q_{T-T_{th}}^*(T_{th} + 1), Q_{T-T_{th}+1}^*(T_{th} + 1)$ is a feasible solution to $\Phi(T_{th})$. We have

$$
\Phi^*(Q_1^*(T_{th} + 1), Q_2^*(T_{th} + 1), \ldots, Q_{T-T_{th}}^*(T_{th} + 1), Q_{T-T_{th}+1}^*(T_{th} + 1))
$$

$$
\leq \Phi^*(Q_1^*(T_{th}), Q_2^*(T_{th}), \ldots, Q_{T-T_{th}+1}^*(T_{th}))
$$

$$
\Rightarrow \sum_{t=1}^{T-T_{th}} (R(Q_t^*(T_{th} + 1)) - C_oQ_t^*(T_{th} + 1)) - C_dQ_{T-T_{th}}^*(T_{th} + 1)
$$

$$
+ R(Q_{T-T_{th}}^*(T_{th} + 1)) - C_oQ_{T-T_{th}}^*(T_{th})
$$

$$
= \Phi^*(T_{th} + 1) + R(Q_{T-T_{th}}^*(T_{th} + 1)) - C_oQ_{T-T_{th}}^*(T_{th} + 1) \leq \Phi^*(T_{th})
$$

$$
\Rightarrow \Phi^*(T_{th}) - \Phi^*(T_{th} + 1) \geq R(Q_{T-T_{th}}^*(T_{th} + 1)) - Q_{T-T_{th}}^*(T_{th} + 1) \geq \rho P^3G(1 - P^3G/R^3G)
$$

$$
\Rightarrow OPT(T_{th}) - OPT(T_{th} + 1) \geq 0.
$$

In summary, $OPT(T_{th}), T_{th} \in [T_1, T_2]$ is a decreasing function of $T_{th}$, so the maximum $OPT(T_{th})$ is achieved when $T_{th} = T_1$.

**Argument 3.** For any $T_{th} \leq T_1$, the maximum objective function value is achieved when $T_{th} = T_1$.

**Proof.** We know that $OPT(T_{th}) - OPT(T_{th} + 1) = \Phi^*(T_{th}) - \Phi^*(T_{th} + 1) - \rho P^3G(1 - P^3G/R^3G), T_{th} \leq T_1$. We only have to prove that $OPT(T_{th}) - OPT(T_{th} + 1) \leq 0, T_{th} \leq T_1$. $Q_2^*(T_{th}), Q_3^*(T_{th}), \ldots, Q_{T+1-T_{th}}^*(T_{th})$ is a feasible solution to $\Phi(T_{th} + 1)$. We have
\[ \Phi^\ast(Q_2^\ast(T_{th}), Q_3^\ast(T_{th}), \cdots, Q_{T+1-T_{th}}^\ast(T_{th})) \leq \Phi^\ast(Q_1^\ast(T_{th}+1), Q_2^\ast(T_{th}+1), \cdots, Q_{T-T_{th}}^\ast(T_{th}+1)) \]

\[ \Rightarrow \sum_{t=1} \left( R(Q_t^\ast(T_{th})) - C_o Q_t^\ast(T_{th}) - C_d Q_{T+1-T_{th}}^\ast(T_{th}) - (R(Q_1^\ast(T_{th})) - C_o Q_1^\ast(T_{th})) \right) \]

\[ = \Phi^\ast(T_{th}) - (R(Q_1^\ast(T_{th})) - C_o Q_1^\ast(T_{th})) \leq \Phi^\ast(T_{th}+1) \]

\[ \Rightarrow \Phi^\ast(T_{th}) - \Phi^\ast(T_{th}+1) \leq R(Q_1^\ast(T_{th})) - C_o Q_1^\ast(T_{th}) < \rho P^{3G}(1 - P^{3G}/R^{3G}) \]

\[ \Rightarrow OPT(T_{th}) - OPT(T_{th}+1) < 0. \]

Similarly, we have the following argument.

**Argument 4.** For any \( T_{th} \in [T_2 + 1, T + 1] \), the maximum objective function value is achieved when \( T_{th} = T + 1 \).

**Proof.** It is easy to see that for any \( T_{th} \in [T_2 + 1, T] \), the maximum objective function value is achieved when \( T_{th} = T \). The proof is similar to the above argument. Then we compare \( OPT(T) \) and \( OPT(T + 1) \).

\[ OPT(T) = S_0 + T \rho P^{3G}(1 - P^{3G}/R^{3G}) + R(Q_1^\ast(T)) - (C_o + C_d)Q_1^\ast(T) < S_0 + (T + 1) \rho P^{3G}(1 - P^{3G}/R^{3G}) = OPT(T + 1). \]

In summary, we have proved that \( \bar{T} = T_1 \) is optimal for \( T_{th} \leq T \).

---

**XI. PROOF OF THEOREM 2**

**Theorem 2.** (Optimal 4G deployment policy with operational cost). The optimal 4G deployment policy is one of the following two options.

- **No deployment scheme:** the operator never deploys any 4G network, i.e., \( Q_t^\ast = 0, t = 1, \ldots, T \).
- **Threshold-based deployment scheme:** when \( t = 1, \ldots, T_{th}^* - 1 \), the operator does not deploy any 4G network, i.e., \( Q_t^\ast = 0 \); when \( t = T_{th}^*, \ldots, T \), the operator deploy 4G network according to Proposition 3 by replacing (3) by (14); and \( T_{th}^\ast \) is determined by Lemma 3.

**Proof.** No deployment scheme corresponds to \( T_{th} = T + 1 \) in Lemma 3. Threshold-based deployment scheme corresponds to \( T_{th} = \bar{T} \) in Lemma 3.
XII. PROOF OF PROPOSITION 5

Proposition 5. The operational cost reduces the final 4G coverage level, and delays the time to deploy the 4G network.

Proof. Without loss of generality, we assume that for both case with/without operational cost, $Q_0 = \Delta P / \Delta R$, so $T_{th} = 0$. Let $Q_T^{op}$ and $Q_{Th}^{nop}$ denote the final coverage level when there is operational cost and when there is no operational cost respectively, $T_{th}^{op}$ and $T_{th}^{nop}$ denote the stopping time when there is operational cost and when there is no operational cost respectively. When there is no operational cost, $Q_T^{op}$ can be reached at time $\tau$, i.e., $Q_{\tau-1}^{nop} \leq Q_T^{op} \leq Q_{\tau-1}^{m,nop}$, and $\tau \leq \hat{T}_{th}^{op}$ because the operator has more capital for investment when there is no operational cost. Now we consider at the stopping time $t = \hat{T}_{th}^{op}$, when there is operational cost.

If $C_d \in [(T-t)(R'(Q_{t-1}^{m}) - C_o), (T-t)(R'(Q_{t-1}^{m}) - C_o)]$, then $C_d - [T - (\tau - 1)]R'(Q_{\tau-1}^{nop}) < C_d - [T - (\tau - 1)]R'(Q_T^{op}) = (\tau - \hat{T}_{th}^{op})R'(Q_T^{op}) - [T - (\hat{T}_{th}^{op} - 1)]C_o < 0$. Hence $\hat{T}_{th}^{nop} \geq \tau$. If $\hat{T}_{th}^{nop} > \tau$, it is obviously that $Q_T^{nop} \geq Q_{\tau-1}^{m,nop} \geq Q_T^{op}$. If $\hat{T}_{th}^{nop} = \tau$, 1) if $C_d < \min\{(T - T_{th}^{nop})R'(Q_{T_{th}^{nop} - 1}^{op}), (T - T_{th}^{nop})R'(Q_{T_{th}^{nop}}^{m,nop})\}$, then $Q_T^{nop} = Q_{\tau-1}^{m,nop} \geq Q_T^{op}$; 2) if $C_d \in [(T - T_{th}^{nop})R'(Q_{T_{th}^{nop} - 1}^{op}), (T - T_{th}^{nop})R'(Q_{T_{th}^{nop}}^{m,nop})]$, then it can be easily derived that $R'(Q_{T_{th}^{nop}}^{m,nop}) < R'(Q_T^{op})$, so $Q_T^{nop} > Q_T^{op}$.

If $C_d > \max\{[(T-t)(R'(Q_{t-1}^{m}) - C_o), (T-t)(R'(Q_{t-1}^{m}) - C_o)]\}$, $\tau \leq \hat{T}_{th}^{op} - 1$ must be true. Because if $\tau = \hat{T}_{th}^{op}$, then $Q_{\tau-1}^{nop} \leq Q_T^{op} = Q_{T_{th}^{op} - 1}^{op} = Q_{\tau-1}^{op}$, but $Q_{\tau-1}^{nop} > Q_{\tau-1}^{op}$ since at the same time period, the achievable coverage when there is no operational cost is strictly higher than that when there is operational cost. $C_d - [T - (\tau - 1)]R'(Q_{\tau-1}^{nop}) < C_d - [T - (\hat{T}_{th}^{op} - 2)](R'(Q_T^{op}) - C_o) < 0$, so $\hat{T}_{th}^{nop} \geq \tau$. If $\hat{T}_{th}^{nop} > \tau$, it is obviously that $Q_T^{nop} \geq Q_{\tau-1}^{m,nop} \geq Q_T^{op}$. If $\hat{T}_{th}^{nop} = \tau$, 1) if $C_d < \min\{(T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop} - 1}^{op}), (T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop}}^{m,nop})\}$, then $Q_T^{nop} = Q_{\tau-1}^{m,nop} \geq Q_T^{op}$; 2) if $C_d \in [(T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop} - 1}^{op}), (T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop}}^{m,nop})]$, then it can be easily derived that $R'(Q_{T_{th}^{nop}}^{m,nop}) < R'(Q_T^{op})$, so $Q_T^{nop} > Q_T^{op}$.

If $C_d < \min\{[(T-t)(R'(Q_{t-1}^{m}) - C_o), (T-t)(R'(Q_{t-1}^{m}) - C_o)]\}$, it is easy to prove that $C_d - [T - (\tau - 1)]R'(Q_{\tau-1}^{nop}) < 0$ and $\hat{T}_{th}^{nop} \geq \tau$. If $\hat{T}_{th}^{nop} > \tau$, it is obviously that $Q_T^{nop} \geq Q_{\tau-1}^{m,nop} \geq Q_T^{op}$. If $\hat{T}_{th}^{nop} = \tau$, 1) if $C_d < \min\{(T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop} - 1}^{op}), (T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop}}^{m,nop})\}$, then $Q_T^{nop} = Q_{\tau-1}^{m,nop} \geq Q_T^{op}$; 2) if $C_d \in [(T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop} - 1}^{op}), (T - \hat{T}_{th}^{nop})R'(Q_{T_{th}^{nop}}^{m,nop})]$, then $R'(Q_{T_{th}^{nop}}^{m,nop}) < R'(Q_T^{op})$, so $Q_T^{nop} > Q_T^{op}$.

\[\square\]
XIII. PROOF OF PROPOSITION 6

Proposition 6. Depending on the 4G coverage $Q_t$, the operator’s revenue is:

- **Low 4G coverage regime.** No users choose 4G service and users with $\alpha \in [\alpha_{(3,0)}, 1]$ choose 3G. The operator’s revenue is:

  $$R(Q_t) = P^{3G}R^{3G} - P^{3G}.$$ 

- **Medium 4G coverage regime.** Users with $\alpha \in [\alpha_{(3,0)}, \alpha_{(4,3)}]$ choose 3G, and $\alpha \in [\alpha_{(4,3)}, 1]$ choose 4G. The operator’s revenue is:

  $$R(Q_t) = P^{3G}\frac{\Delta P R^{3G} - \Delta R P^{3G} + \gamma(1 - Q_t)(P^{3G} - R^{4G} + \Delta P)}{\Delta R R^{3G} + \gamma(R^{3G} Q_t + \Delta R)} + P^{4G}\frac{\Delta R R^{3G} - \Delta P R^{3G}}{\Delta R R^{3G} + \gamma(R^{3G} Q_t + \Delta R)} + (1 - Q_t)(P^{3G} - \Delta P/Q_t).$$

- **High 4G coverage regime.** No users choose 3G service and users with $\alpha \in [\alpha_{(4,0)}, 1]$ choose 4G. The operator’s revenue is:

  $$R(Q_t) = P^{4G}\frac{Q_t \Delta R + R^{3G} - P^{4G}}{Q_t \Delta R + R^{3G} + (1 - Q_t)^2\gamma}.$$ 

Proof. Without confusion, here to ignore the subscript $t$ for clarity of expression. Let $D^{3G}$ and $D^{4G}$ denote the number of users who choose 3G or 4G service respectively. Let $D = \gamma(D^{3G} + (1 - Q)D^{4G})$.

- **Low 4G coverage regime,** as shown in Fig.1(a). We see that $\alpha_{(4,3)} > 1$, so $Q < \frac{\Delta P R^{3G} + \gamma \Delta P}{\Delta R R^{3G} + \gamma R^{3G} + \gamma P^{3G}}$,

  \begin{align*}
  D^{4G} &= 0 \\
  D^{3G} &= \frac{R^{3G} - P^{3G}}{R^{3G} + \gamma}
  \end{align*}

- **Medium 4G coverage regime,** as shown in Fig. 1(b). The condition is $\alpha_{(3,0)} < \alpha_{(4,0)} < \alpha_{(4,3)}$, that is, $Q < \Delta P R^{3G}/(P^{3G} \Delta R + R^{4G} D)$. We can derive that
\[ D = \frac{\gamma(\Delta PR^3G - \Delta RP^3G + (1 - Q)R^3G\Delta R)}{\Delta RR^3G + \gamma(R^3GQ + \Delta R)} \]

\[ D^3G = \frac{\Delta PR^3G + \gamma(1 - Q)\Delta P + Q\gamma(1 - Q)P^3G - \gamma(1 - Q)QR^4G - Q\Delta RP^3G}{Q[\Delta RR^3G + \gamma(R^3GQ + \Delta R)]} \]

\[ D^4G = \frac{Q(\Delta RR^3G + \gamma R^4G - \gamma P^3G) - \Delta PR^3G - \gamma \Delta P}{Q[\Delta RR^3G + \gamma(R^3GQ + \Delta R)]} \]

Now we can get the close form expression of the condition \( Q < \Delta PR^3G/(P^3G\Delta R + R^4G D) \).

We get the following inequality.

\[ \gamma(R^4G - P^3G)Q^2 - [\Delta RP^3G + \gamma(\Delta P - P^3G + R^4G)]Q + \Delta P(R^3G + \gamma) > 0 \]

Define

\[ M_1 = \gamma(R^4G - P^3G) \]

\[ M_2 = \Delta P(R^3G + \gamma) \]

\[ M_3 = \Delta RP^3G - \Delta PR^3G > 0 \]

So we have

\[ M_1Q^2 - (M_1 + M_2 + M_3)Q + M_2 > 0 \]

\[ Q > Q_1 = \frac{M_1 + M_2 + M_3 + \sqrt{(M_1 - M_2 + M_3)^2 + 4M_2M_3}}{2M_1} \]

or \( Q < Q_2 = \frac{M_1 + M_2 + M_3 - \sqrt{(M_1 - M_2 + M_3)^2 + 4M_2M_3}}{2M_1} \)

We can easily prove that \( Q_1 > 1 \). So the reasonable one is \( \frac{\Delta PR^3G + \gamma \Delta P}{\Delta RR^3G + \gamma R^4G - \gamma P^3G} \leq Q < Q_2 \).

- **High 4G coverage regime**, as shown in Fig. 1(c). The condition is \( \alpha_{(3,0)} > \alpha_{(4,0)} > \alpha_{(4,3)} \), that is \( Q_2 \leq Q \leq 1 \). We can derive the user distribution as follows:

\[ D^3G = 0 \]

\[ D^4G = \frac{QR^4G + (1 - Q)R^3G - P^4G}{(QR^4G + (1 - Q)R^3G) + (1 - Q)^2\gamma} \]